Amortization

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An amount of money P_0 is borrowed for a period of y years at an annual interest rate r. A monthly payment of M is made over the period of the loan. This situation can easily be described by a non-homogeneous linear recurrence relation.

Let P_n denote the balance ("principal") still owing at the beginning of the $n+1^{\text{st}}$ month, just after the n^{th} monthly payment has been made. The relationship between P_n and P_{n-1} is as follows:

Tt the beginning of the $n+1^{\text{st}}$ monthly, the previous balance P_{n-1} is increased by the interest owing for the month, namely $\dfrac{r}{12}P_{n-1}$, and decreased by the amount of the monthly payment M. Thus the equation relating P_{n-1} and P_n is

$$P_n = P_{n-1} + \frac{r}{12}P_{n-1} - M.$$

Thus we have a **non-homogeneous** linear recurrence relation

$$P_n - \left(1 + \frac{r}{12}\right) P_{n-1} = -M.$$

To solve this, we first look for a particular solution. Since the right-hand side is a constant, we try $P_n = A$, where A is a constant to be determined.

Substituting back into the recurrence relationship shows that we should choose $P_n = \dfrac{12M}{r}$.

Intuitively this makes sense. For if the amount borrowed is $\dfrac{12M}{r}$, then each month the interest owing will be exactly equal to the monthly payment, so that the balance owed on the loan will in fact remain constant.

Now since the solution to the corresponding homogeneous recurrence relation

$$P_n - \left(1 + \frac{r}{12}\right) P_{n-1} = 0$$

is $P_n = C\left(1+\frac{r}{12}\right)^n$, the **general solution** to the original recurrence relation is

$$P_n = \frac{12M}{r} + C\left(1+\frac{r}{12}\right)^n,$$

where C is an arbitrary constant.

To make this agree with the given initial balance P_0, we require that $P_0 = \frac{12M}{r} + C$, so that $C = \left(P_0 - \frac{12M}{r}\right)$, giving the solution

$$P_n = \frac{12M}{r} - \left(\frac{12M}{r} - P_0\right)\left(1+\frac{r}{12}\right)^n.$$

This formula allows one to determine the balance owing after the n^{th} month, assuming that one knows the initial amount borrowed and the monthly payment M.

However in practice, one is often interested in computing what the monthly payment should be in order for the loan to be paid off after y years. Rephrased, this question says: What should M be in order that $P_{12y} = 0$?

Substitutating into the equation, one has

$$P_{12y} = \frac{12M}{r} - \left(\frac{12M}{r} - P_0\right)\left(1+\frac{r}{12}\right)^{12y},$$

so setting $P_{12y} = 0$, one gets

$$0 = \frac{12M}{r} - \left(\frac{12M}{r} - P_0\right)\left(1+\frac{r}{12}\right)^{12y},$$

$$\text{so that} \quad \frac{12M}{r}\left[\left(1+\frac{r}{12}\right)^{12y} - 1\right] = P_0\left(1+\frac{r}{12}\right)^{12y},$$

$$\text{and thus} \quad M = \frac{r}{12} P_0 \left(1+\frac{r}{12}\right)^{12y} \Big/ \left[\left(1+\frac{r}{12}\right)^{12y} - 1\right].$$