MATH 611 **Due Wed Oct 21, 1992**

1. Prove that if a module M satisfies the maximum principle then every submodule of M is finitely generated.

2. Let R be the ring $K[X_1, X_2, \ldots]$ of polynomials in the countably many variables X_i, $i = 1, 2, 3, \ldots$ Let I be the ideal generated by all the variables X_i. (One can also describe I as consisting of all those polynomials with trivial constant term.) Prove that I is not a finitely generated ideal.

3. A family of submodule $\{M_i\}_{i \in I}$ is called **directed** if for each pair $i, j \in I$ there exists $k \in I$ such that $M_i, M_j \subseteq M_k$. Prove that if $\{M_i\}_{i \in I}$ is a directed family of submodules of M then $\bigcup_I M_i$ is a submodule of M.

4. Let K be a field and let R be the ring of two-by-two upper triangular matrices $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$ with entries in K. Write $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\mathbf{j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

a) Let M be an R-module . Prove that $\mathbf{j}M = \mathbf{j}(\mathbf{e}_2 M)$ and that $\dim \mathbf{j}M \leq \min\{\dim \mathbf{e}_1 M,\ \dim \mathbf{e}_2 M\}$. (Here $\mathbf{j}M = \{\mathbf{j}m \mid m \in M\}$ and $\dim X$ means the dimension of X considered as a vector space over K. If you need to, you can consider only the case where these dimensions are finite.)

b) Prove that if U is a K-subspace of M (considering M as a vector space over K), then U is an R-submodule of M if and only if $\mathbf{e}_1 U \subseteq U$, $\mathbf{e}_2 U \subseteq U$, and $\mathbf{j}U \subseteq U$.

c) Prove that if r, s, or t are cardinal numbers (or positive integers, if you don't understand cardinals) such that $t \leq \min\{r, s\}$, then there exists an R-module M such that $\dim \mathbf{e}_1 M = r$, $\dim \mathbf{e}_2 M = s$, and $\dim \mathbf{j}M = t$.

d) Prove that if M and N are R modules, then M and N are isomorphic if and only if $\dim \mathbf{e}_1 M = \dim \mathbf{e}_1 N$, $\dim \mathbf{e}_2 M = \dim \mathbf{e}_2 N$, and $\dim \mathbf{j}M = \dim \mathbf{j}N$.

e) Consider the three following R-modules : $K \oplus 0$, $0 \oplus K$, and $K \oplus K$, where in each case the first summand equals $\mathbf{e}_1 M$ and the second equals $\mathbf{e}_2 M$; for the first two modules, multiplication by $\mathbf{j}$ is defined to be trivial, and in the third case it corresponds to the identity map from K to K. (E.g. the R-multiplication on $K \oplus K$ is determined by the rules $\mathbf{e}_1(x, y) = (x, 0)$, $\mathbf{e}_2(x, y) = (0, y)$ and $\mathbf{j}(x, y) = (y, 0)$.) Prove that every R-module is isomorphic to a direct sum of copies of these three modules.

4. **a)** Simple computation shows that $\mathbf{j} = \mathbf{j}\mathbf{e}_2$. Then $\dim \mathbf{j}M \leq \dim \mathbf{e}_1 M$ because $\mathbf{j}M \subseteq \mathbf{e}_1 M$, and $\dim \mathbf{j}M \leq \dim \mathbf{e}_2 M$ because $\mathbf{j}M = \mathbf{j}(\mathbf{e}_2 M)$ and multiplication by $\mathbf{j}$ is a linear transformation, so the dimension of the image is at most the dimension of the domain.

b) "$\Rightarrow$" is self-evident, and "$\Leftarrow$" follows from the fact that every $r \in R$ has the form $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + k_3\mathbf{j}$ with $k_i \in K$.

c) We start with vector spaces V and W with $\dim V = r$, $\dim W = s$. Construct a linear transformation $\theta\colon W \longrightarrow V$ such that $\dim \theta(W) = t$. (The way to construct such an θ is by considering a basis for W. Since W is a free K-module, we know from a theorem proved in class that there exists a linear transformation θ sending the basis elements of W to any elements we choose in V.) Now represent elements of $V \oplus W$ by **column** vectors $\begin{pmatrix} v \\ w \end{pmatrix}$. We can make $V \oplus W$ into an R-module by defining the scalar multiplication as follows:

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a_{11}v + a_{12}\theta(w) \\ a_{22}w \end{pmatrix}.$$

Using the rules for matrix multiplication, one can easily check that the axioms for an R-module are satisfied.

d) "$\Rightarrow$" is routine, since if φ is an isomorphism from M to N, then φ restricts to isomorphisms from $\mathbf{e}_1 M$, $\mathbf{e}_2 M$, and $\mathbf{j}M$ **onto** $\mathbf{e}_1 N$, $\mathbf{e}_2 N$, and $\mathbf{j}N$ respectively (WHY?).

Now suppose the dimensions match. We will construct an isomorphism φ from M onto N. The difficult part will be to make sure it is R-linear.

Choose a basis for M as a vector space in the following way: Choose a basis $m_1, ..., m_r \in \mathbf{e}_2 M$ for the kernel of the linear transformation $\mathbf{e}_2 M \longrightarrow \mathbf{e}_1 M$ given by $x \mapsto \mathbf{j}x$. (The notation suggests that r is finite, but this is not really essential to the proof.) Extend this to a basis for $\mathbf{e}_2 M$ by choosing additional basis elements $m_{r+1}, ..., m_s$. Then $s - r = \dim \mathbf{j}M$ and the elements $\mathbf{j}m_{r+1}, ..., \mathbf{j}m_s$ form a linearly independant set in $\mathbf{e}_1 M$ (WHY?). In fact, $m_1, ..., m_r, ..., m_s, \mathbf{j}m_{r+1}, ..., \mathbf{j}m_s$ form a linearly independant set in M (WHY?). Finally, choose additional elements $m_{s+1}, ..., m_t$ in $\mathbf{e}_1 M$ so that

$$m_1, ..., m_r, m_{r+1}, ..., m_s, \mathbf{j}m_{r+1}, ..., \mathbf{j}m_s, m_{s+1}, ..., m_t$$

is a basis of M as a K-vector space. (*Why is this possible?*) Then choose a corresponding basis $n_1, ..., n_r, ..., n_s, \mathbf{j}n_{r+1}, ..., \mathbf{j}n_s, ..., n_t$ in exactly the same way for N. The reason we can get the subscripts for the m's and n's to match exactly is because of the hypothesis on the various dimensions.

We know that we can define a K-linear transformation $\varphi\colon M \longrightarrow N$ by specifying where basis elements go. So choose $\varphi(m_i) = n_i$ and $\varphi(\mathbf{j}m_i) = \mathbf{j}n_i$ for all values of i such that this makes sense. Then φ is an isomorphism of vector spaces because it maps one basis to another. The crucial part is to see that φ is R-linear, and for this it suffices to see that $\varphi(\mathbf{e}_i m) = \mathbf{e}_i\varphi(m)$ and $\varphi(\mathbf{j}m) = \mathbf{j}\varphi(m)$ for all $m \in M$. Since $\mathbf{e}_i^2 = \mathbf{e}_i$, it can easily be seen that $\varphi(\mathbf{e}_i m) = \mathbf{e}_i\varphi(m)$ for all $m \in M$ if and only if $\varphi(\mathbf{e}_i M) \subseteq \mathbf{e}_i M$ for $i = 1, 2$. This is clear from our construction of φ. Furthermore, one easily sees (!) that it suffices to check the equation $\varphi(\mathbf{j}m) = \mathbf{j}\varphi(m)$ for basis elements. But this is clear from the construction of φ, if one remembers that $\mathbf{j}m_i = 0$ for $i = 1, ..., r$ and $i = s+1, ..., t$, and likewise for the n_i.

e) It is perhaps less confusing to denote the three modules in question by $\begin{pmatrix} K \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ K \end{pmatrix}$, and $\begin{pmatrix} K \\ K \end{pmatrix}$. For $\begin{pmatrix} K \\ 0 \end{pmatrix}$ and $\begin{pmatrix} K \\ K \end{pmatrix}$, the scalar multiplication is just ordinary matrix multiplication:

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{22}y \end{pmatrix}.$$

For $\begin{pmatrix} 0 \\ K \end{pmatrix}$ the multiplication is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ a_{22}y \end{pmatrix}.$$

(NOTE: The set L of matrices of the form $\begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}$ is a left ideal (in fact, two-sided) in R. It is easy to see that $\begin{pmatrix} 0 \\ K \end{pmatrix}$ is isomorphic to R/L.)

Now let M be an R-module and choose a basis for the vector space M in the same form as in part **d)**. For $i = 1, ..., r$, let M_i be the one-dimensional K-vector space generated by m_i. Since by assumption $m_i \in \mathbf{e}_2 M$ and $\mathbf{j}m_i = 0$, we easily check that $\mathbf{e}_k M_i \subseteq M_i$ and $\mathbf{j}M_i \subseteq M_i$, so that M_i is an R-submodule of M. Furthermore $M_i \approx \begin{pmatrix} 0 \\ K \end{pmatrix}$. Likewise construct M_i for $i = s+1, ..., t$ as the one-dimensional K-space generated by the corresponding m_i and notice that these M_i are also R-submodules of M and $M_i \approx \begin{pmatrix} K \\ 0 \end{pmatrix}$. Finally, for $i = r+1, ..., s$, let M_i be the two dimensional K-space generated by m_i and $\mathbf{j}m_i$. Then $\mathbf{e}_i M_i \subseteq M_i$ and $\mathbf{j}M_i = \mathbf{e}_1 M_i \subseteq M_i$, so these are also R-submodules of M and in this case $M_i \approx \begin{pmatrix} K \\ K \end{pmatrix}$.

It should be clear that M is the direct sum of all these M_i. (NOTE: Obviously we could have derived **d)** from **e)**.