THE JACOBSON RADICAL OF A RING

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THEOREM. Let R be a non-trivial ring and $r \in R$. The following conditions are equivalent and the set of $r \in R$ satisying these conditions forms a (two-sided) ideal.

(1) $rM = 0$ for all simple left R-modules M.
(2) $Nr = 0$ for all simple right R-modules N.
(3) $\varphi(r) = 0$ for all homomorphisms φ from R into a simple left R-module.
(4) $\psi(r) = 0$ for all homomorphisms ψ from R into a simple right R-module.
(5) r belongs to all maximal left ideals in R.
(6) r belongs to all maximal right ideals in R.
(7) For all $x \in R$, $1 - xr$ is invertible.
(8) For all $x \in R$, $1 - rx$ is invertible.

PROOF: Let J denote the set of elements satisfying condition (1). It is easy to see that J is a two-sided ideal.

(1) $\Rightarrow$ (3): Let $\varphi\colon R \to M$, where M is a simple left R-module. If $rM = 0$ then $\varphi(r) = r\varphi(1) = 0$.

(3) $\Rightarrow$ (1): Let M be a simple R-module and $m \in M$. There exists an R-linear map φ $R \to M$ given by $\varphi(x) = xm$. Then $rm = \varphi(r)$ so if $\varphi(r) = 0$ for all $\varphi\colon R \to M$ then $rm = 0$ for all $m \in M$, i. e. $rM = 0$.

(3) $\Leftrightarrow$ (5): If M is a simple left R-module and $\varphi\colon R \to M$ is non-trivial then $\operatorname{Ker}\varphi$ must be a maximal left ideal L (WHY?). On the other hand, if L is a maximal left ideal and $\varphi\colon R \to R/L$ is the quotient map then φ is a map from R into a simple left R-module and $\operatorname{Ker}\varphi = L$. Thus $\varphi(r) = 0$ for all maps φ from R into simple left R-modules if and only if r belongs to all maximal left ideals.

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6): Analogous.

(5) $\Rightarrow$ (7): If r belongs to every maximal left ideal then so does xr for any $x \in R$. It follows easily that $1 - xr$ does not belong to any maximal left ideal and therefore $1 - xr$ must be left invertible. Let $s \in R$ be such that $s(1 - xr) = 1$. Then $1 - s = -sxr \in J$ so by the preceding $s = 1 - (1 - s)$ is left invertible, say $ts = 1$. But then $t = t1 = ts(1 - xr) = 1(1 - xr) = 1 - xr$, so $s(1 - xr) = 1 = ts = (1 - xr)s$ and $1 - xr$ is in fact invertible.

(7) $\Rightarrow$ (5): Let L be a maximal left ideal. If $1 - xr$ is invertible for all $x \in R$ then $1 - xr \notin L$. Thus $1 \notin Rr + L$, so $L \subseteq Rr + L \subsetneq R$. Since L is a maximal left ideal, this forces $L = Rr + L$, and therefore $r \in L$.

(6) $\Leftrightarrow$ (8): Analogous.

(7) $\Rightarrow$ (1): [ALTERNATE PROOF] Suppose that $1 - xr$ is invertible for all $x \in R$ and suppose BWOC that $rM \neq 0$ for some simple left R-module M. Choose $m \in M$ with $rm \neq 0$. Then Rrm is a non-trivial submodule of M so $M = Rrm$ since M is simple. Thus $m \in Rrm$ so $m = xrm$ for some $x \in R$. Thus $(1 - xr)m = 0$. But by assumption there exists $u \in R$ such that $u(1 - xr) = 1$. Therefore $m = u(1 - xr)m = 0$, a CONTRADICTION. [This is a special case of Nakayama's Lemma.]

(7) $\Rightarrow$ (8): Since we have seen that (1) is equivalent to (7), the set of elements satisfying (7) is the two-sided ideal J. Thus if r satisfies (7) then so does rx for any x, so by (7) $1 - rx$ is invertible for all x.

(8) $\Rightarrow$ (7): Analogous. $\boxed{\checkmark}$