HOW TO USE INTEGRALS

E. L. Lady

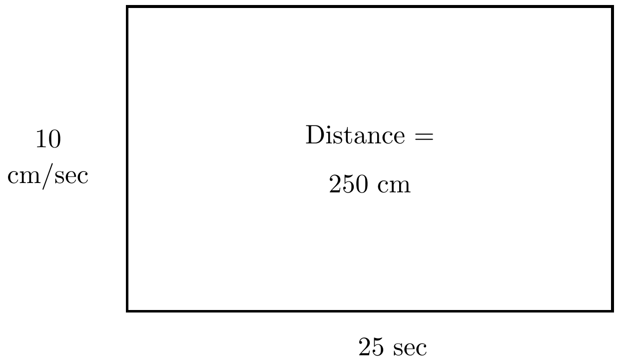
(December 21, 1998)

Consider the following set of formulas from high-school geometry and physics:

Area	=	Width × Length	Area of a Rectangle
Distance	=	Velocity × Time	Distance Traveled by a Moving Object
Volume	=	Base Area × Height	Volume of a Cylinder
Work	=	Force × Displacement	Work Done by a Constant Force
Force	=	Pressure × Area	Force resulting from a constant pressure
Mass	=	Density × Volume	Mass of a solid with constant density.

There's a common simple pattern (structure) to all these formulas, namely $A = BC$.

The first four of these formulas can be interpreted graphically. Namely, in the second formula, for instance, if one constructs a rectangle, where the vertical side corresponds to a velocity V and the horizontal side to time T, then the area of this rectangle will represent the distance traveled by an object moving at velocity V for a time T.

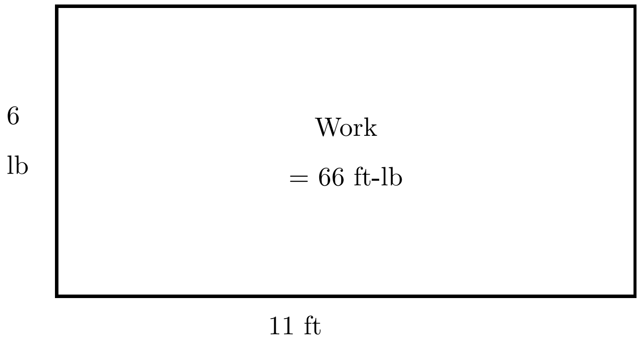


(Notice that since horizontal units in this picture represent seconds, and vertical units represent cm/sec, the units for the area of the rectangle should be

$$\text{sec} \times \frac{\text{cm}}{\text{sec}} = \text{cm},$$

as shown.

Likewise, the fourth formula could be represented graphically by a rectangle where the horizontal side represents distance (displacement) and the vertical side represents force. Once again, the area of the rectangle represents the work done by the moving force.



A force of 6 pounds is applied to an object which moves for a distance of 11 feet.

(The last two formulas could also be represented by rectangles, but in this case the horizontal side of the rectangle would be a one-dimensional representative area or volume. In these cases, it's more useful to represent the equation by a three or four-dimensional figure rather than by a rectangle.)

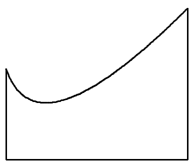
The Need for Integrals

All six of the formulas above are simplistic. In the real world, objects don't travel at constant speed. They speed up, slow down, even stop for a while to rest or get fuel. To have a really useful formula relating velocity and distance, we need to consider the possibility that velocity v is not a constant but instead is a function of time t.

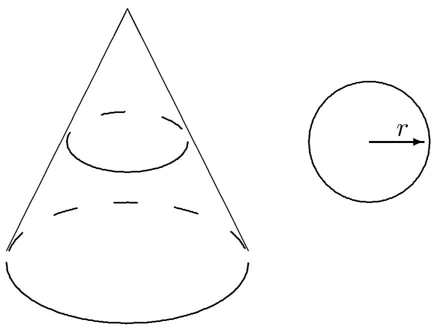
Likewise, instead of the third formula, giving the volume of a cylinder, it would be more useful to consider a solid where the horizontal cross section changes as the height changes, i. e. the area of the cross section at height h is a function $A(h)$ of h.

And in the remaining examples, it is important to deal with situations where force, pressure, or density are variable.

And even in case of the first formula, we realize that not all plane regions are rectangles, so it would be useful to have a formula for the area of a region where the width is a variable function of the horizontal position.



A "rectangle" with a deformed top



In this cone, the horizontal cross section at height h is a circle with radius $r = 8 - 2h$, where h ranges from 0 at the bottom to $h = 4$ at the top. The volume of the cone turns out to be given by the formula $Vol = \int_0^4 \pi(8-2h)^2\,dh$.

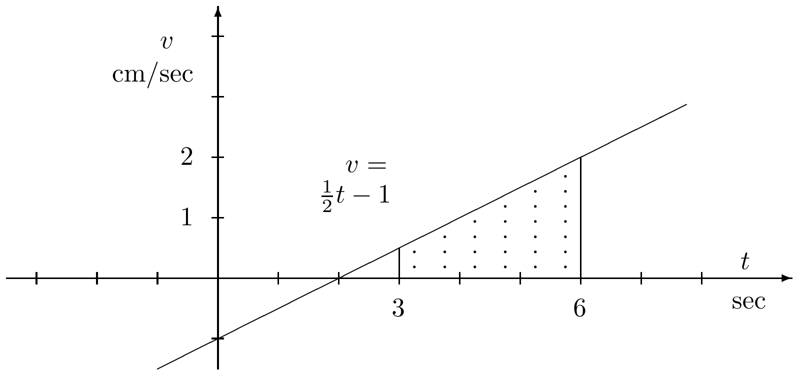
More Realistic Problems

Consider now a simple, but slightly more realistic, time-velocity problem:

► The velocity of an object between time $t = 3$ sec and $t = 6$ sec is given by the formula $v(t) = \frac{1}{2}t - 1$ (measured in units of cm/sec). How far does the object travel during this time period?

What this problem is more or less asking is: *How does one multiply velocity by elapsed time in a case when velocity is actually changing as time progresses?*

We have seen above that area can be a geometric way of representing multiplication. Now represent the above problem by graphing the function $v = \frac{1}{2}t - 1$.



It turns out that the distance traveled by the object between time $t = 3$ and time $t = 6$, as described in the problem, will be equal to the area over that part of the t-axis between $t = 3$ and $t = 6$ and under the graph of the function $v = 2t - 1$, as indicated by the shaded region in the graph.

Note that in the graph, the horizontal units represent seconds and the vertical units are measured in cm/sec. Logically, then, the units for area should be

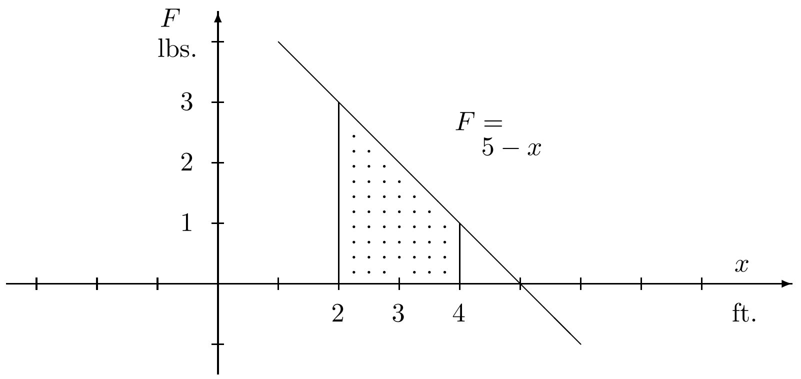
$$\frac{\text{cm}}{\text{sec}} \times \text{sec} = \text{cm}\,,$$

as required.

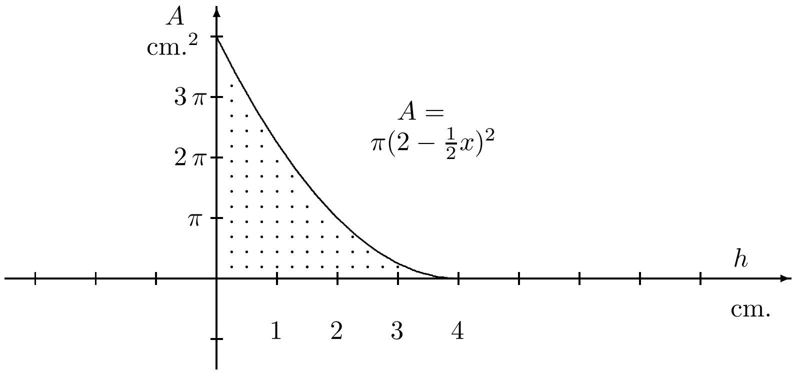
Likewise, consider a force-displacement-work problem.

► A vertical force on an object moving horizontally is given by the formula $F(x) = 5 - x$ (where x is measured in feet and F is measured in pounds). Find the work done by this force while the object moves between the points $x = 2$ ft and $x = 4$ ft.

Analogously to the velocity-time problem, we can "multiply" the variable force in this problem by the displacement, by measuring the area under the graph of the function $F(x) = 5 - x$ between the points $x = 2$ and $x = 4$, as indicated on the graph below. (The units for work in this case are ft-lbs.)



Similarly, if one constructs the graph of the cross-sectional area of a solid, where the horizontal coordinate h represents height and the vertical coordinate represents the area $A(h)$ of the horizontal cross-section of the solid at that height, then the volume of that solid will be given by the area between this curve and that part of the h-axis between the bottom height (h_{start}) and the height at the top (h_{end}). (In this case, no part of the curve will be below the h-axis, since the cross-sectional area is never negative.)



The area of the horizontal cross section of a cone at height h equals $\pi(2-\frac{1}{2}h)^2$ cm^2. The volume of the cone will equal the area under the curve $A=\pi(2-\frac{1}{2}h)^2$ for h between 0 and 4.

In the language of calculus, the six simplistic high-school formulas at the beginning of these notes are replaced by formulas given by integrals.

$$A = \int_{x_{\text{start}}}^{x_{\text{end}}} w(x)\,dx$$

$$D = \int_{t_{\text{start}}}^{t_{\text{end}}} v(t)\,dt$$

$$V = \int_{h_{\text{start}}}^{h_{\text{end}}} A(h)\,dh$$

$$W = \int_{x_{\text{start}}}^{x_{\text{end}}} F(x)\,dx$$

$$F = \iint_{\Omega} p(x,y)\,dx\,dy$$

$$M = \iiint_{T} \rho(x,y,z)\,dx\,dy\,dz.$$

There are many other problems where this same idea applies. In order to be able to tell whether it applies to a given situation, we are going to consider below the reasons why it is true.

In physics books, concepts such as work are often simply *defined* by formulas in the form of integrals: Work $= \int_a^b F(x)\,dx$. This neatly sidesteps the need to prove that these formulas are correct. However since such definitions are, for most students, extremely non-intuitive, one might wonder to what extent they are arbitrary. Could one also get a satisfactory theory by using some completely different definition for work? In fact, from the principles to be given below, one can see that even before one knows how to formally define the concept of work, the formula Work $= \int_a^b F(x)\,dx$ can be seen to be an essential consequence of simple axioms about the relationship between force and work that almost everyone will accept as self-evident. Namely, it follows from the fact that the relationship between force and work is (using words that will be defined below) *cumulative* and *increasing.*

What Makes a Relationship Expressible by an Integral?

The purpose of these notes is to give two simple principles which will enable one in most cases to recognize when a mathematical relationship can be expressed in terms of an integral and to be able to set up the correct integral with a fair amount of confidence.

Before explaining these principles, it will be useful to note several examples of formulas in physics and other sciences where the basic pattern $A = BC$ is valid even without simplistic assumptions and does **not** generalize to a formula given by an integral. Namely, consider Ohm's Law relating voltage, current and resistance; Newton's Second Law of Motion ($F = ma$); Hook's Law for the force exerted by a stretched (or compressed) spring; and the Boyle-Charles Law relating volume, temperature, and pressure for an ideal gas.

Some Formulas Which Never Generalize to Integrals

$E = IR$	(Ohm's Law for voltage, current and resistance.)
$F = ma$	(Newton's Second Law of Motion.)
$W = EI$	(Electrical power is the product of current & voltage.)
$T = kPV$	(The Boyle-Charles Law for temperature, pressure, and volume.)
$F = kx$	(Hook's Law for springs.)

Betweenness

One of the most important things that makes the formulas

$$\begin{aligned} \text{Area} &= \text{Height} \times \text{Length} \\ \text{Distance} &= \text{Velocity} \times \text{Time} \\ \text{Volume} &= \text{Base Area} \times \text{Height} \\ \text{Work} &= \text{Force} \times \text{Displacement} \end{aligned}$$

different from formulas such as Ohm's Law and Hook's Law is that these four formulas all have the property that the second factor on the right actually represents the amount by which a certain variable changes. For instance, the second formula could be better written as

$$\text{Distance} = \text{Velocity} \times \text{Elapsed Time}$$

or

$$\text{Distance} = \text{Velocity} \times \text{Time Interval}.$$

Likewise, in the formula for the volume of a cylinder, the factor called "height" is actually the difference in height between the top and the bottom of the cylinder. And in the fourth formula, for work, "displacement" is the measure of a change in position. And the same thing is even true of "length" in the formula for the area of a rectangle, i. e. "length" is the difference between the position of the right edge of the rectangle and the position of the left edge.

If we now use the variable x to represent horizontal position, y to represent vertical position, and t to represent time, then the four above formulas could be expressed as, using rather obvious notion,

$$\begin{aligned} A &= (x_{\text{end}} - x_{\text{start}})W \\ D &= (t_{\text{end}} - t_{\text{start}})V \\ V &= (y_{\text{end}} - y_{\text{start}})A \\ W &= (x_{\text{end}} - x_{\text{start}})F. \end{aligned}$$

By contrast, in Ohm's Law $E = IR$, the voltage E is determined purely by the size of the resistance and current at a given instant, and is not influenced by any other values that resistance and current might have taken in the past, so that a formula given by an integral would not be appropriate. If we try to imagine a generalization of Ohm's Law written in the form of an integral,

$$E = \text{Voltage} = \int_{?}^{?} R(i)\, di\,,$$

(where i represents current and $R(i)$ resistance), there is no reasonable choice for what values to put at the top and the bottom of the integral sign, since the current I cannot naturally thought of as the amount of change made by some variable. Furthermore, the notation $R(i)$ is inappropriate since resistance is not normally a function of current.

One might say that in those situations where high-school formulas generalize to integrals there is a notion of *betweenness.* A moving object travels *between* a starting time and an ending time, and to know how fast it travels we need to know the velocity at *all* the instants *between* these two times. Likewise, a solid body has horizontal cross-sections at all heights between some starting height h_0 and some ending height h_1, and we can compute the volume of the solid if we know the area of these cross sections at *all* the heights *between* these two.

As a rule of thumb, any time one of the two factors on the right-hand side of a formula $A = BC$ represents time or distance, one can suspect that the formula corresponding to the general situation will be given by an integral. (Note that Hook's Law for springs, mentioned above, $F = -kx$, is one exception. Even if imagines a situation where the spring constant k would be a function of x, Hook's Law would still not be given by an integral, because the force exerted by the spring would still depend only on the length x to which the spring has been stretched (or compressed), and not by anything that happens at points in between the spring's rest point and x.)

Cumulative Relationships

I think that it is pretty fair to say that if one *thinks* that a mathematical relationship might have some expression in the form of an integral, then there will in fact exist an integral expressing that relationship. The problem then becomes to find the correct integral formula, which can be more difficult.

A more sophisticated criterion for the existence of an integral expressing a given relationship involves the notion of a **cumulative** relationship. An example will make the idea clear.

The relationship between velocity and distance is cumulative in that if one considers a time t_2 in between two other times t_1 and t_3, then the distance an object travels between times t_1 and t_3 can be obtained by adding together the distance traveled between time t_1 and time t_2 plus the distance traveled between t_2 and t_3.

SIDEBAR: What Is an Integral?

In beginning calculus courses, the integral is introduced by discussing the problem of finding the area under a curve. Dividing the area into tiny vertical strips, one arrives at the concept of a Riemann sum (or some variation on this idea). A theorem is then proved stating that under reasonable conditions such a Riemann sum will converge to a limit as the width of the rectangles goes to zero. The integral is then defined to be the limit guaranteed to exist by this theorem.

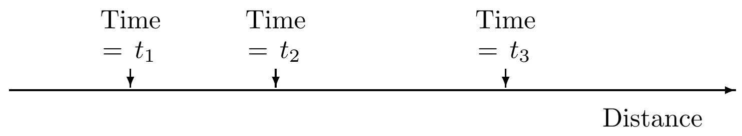
This seems to say that the way to find the area under a curve involves a monstrosity that apparently no one could ever compute in practice.

There's a point here that mathematicians take for granted, but students are often not explicitly told. Namely, it doesn't matter if the definition of the integral is completely impractical, because the definition of a concept doesn't have to be something one actually uses – except to prove a few theorems. *In mathematics, it's not important what things are. What's important is how they behave — the rules they obey.* The definition of a concept is simply a way of getting your foot in the door. It gives you a firm foundation to develop the rules which the concept obeys and which are the things that everybody really uses in practice. (In many cases, such as with the integral, what a definition really is is an existence theorem.) In practice, integrals can be computed by anti-derivatives, so that the Riemann sums are for the most part irrelevant.

The point of view of these notes is to encourage students to think of the integral of a function as the area under its curve between the two prescribed points (with the added proviso that area below the x-axis should be considered negative). (The graph of a function of two variables is of course a surface, and a double integral is equal to the volume under that surface. Unfortunately, triple integrals are difficult to visualize in an analogous way.)

As a mathematician, however, I feel compelled to point out that defining the integral as the area under a graph ultimately doesn't simplify things at all. This is because giving a rigorous mathematical definition of area is not any easier (nor much more difficult) than developing the integral on the basis of Riemann sums.

Secondly, there exist functions so unruly that their graphs are extremely disconnected and don't even look like coherent curves, so that it doesn't make any sense to talk about the area under these graphs. (The idea is a little like that of a fractal curve. But instead of radically changing direction infinitely often like a fractal, these curves have discontinuous breaks infinitely often.) Fortunately, however, beginning calculus students (and most people who use calculus as a tool) don't have to deal with such functions and can manage quite well by depending on their intuitive ideas about area.



Likewise, the relationship between pressure and force is cumulative. If a function $p(x, y)$ describes a pressure applied to a certain planar region, and if one considers two non-overlapping (two-dimensional) pieces of that region, then the force exerted by that pressure on the union of the two pieces will be the sum of the forces exerted on each piece.

One rough, informal, non-technical definition of the integral is that $\int_a^b f(x)\,dx$ gives the **cumulative effect** of the function $f(x)$ when applied to all the values of x between a and b. For instance, the cumulative effect when a velocity function $v(t)$ is applied at all the moments of time t between $t = a$ and $t = b$ will be the distance an object whose velocity is given by that function would travel.

As a practical matter, almost any time a scientific relationship between a quantity and the values of a function over an interval has the property of betweenness, then that relationship will be cumulative. A relationship which is not cumulative would be roughly comparable to real-life situations such as air travel, where the time to travel between New York and Los Angeles would not be the sum of the time to travel from New York to Chicago and the time from Chicago to Los Angeles (assuming the first flight was non-stop).

I can now present the main point of these notes, namely two rules of thumb for expressing a mathematical relationship in the formula. This is as far in the article as many people will need to read.

Two Rules of Thumb

(1) In general, almost any time a quantity is determined by the values of a function over an interval in a cumulative way, one can be fairly certain that the relationship between the quantity and the function be expressed as an some sort of integral.

(2) In most cases, if it is known that the relationship between a quantity and a function is expressible by some integral, and if a suggested integral formula for this relationship yields the correct answers in cases when the function is a constant, then the formula will be correct.

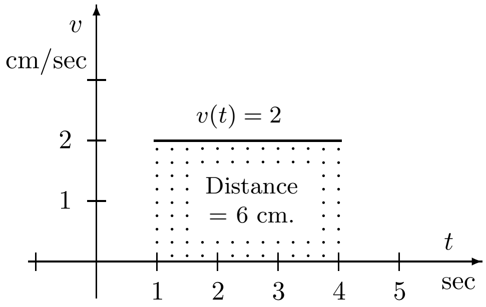
The first rule of these rules of thumb is more universally valid than the second. To enable students to understand when the second rule of thumb will apply, I will first discuss the reasons why it works, and the basic assumptions involved. After this, I will discuss one example of a situation where the second principle fails: namely, the formula for the length of a curve.

Why The Second Rule of Thumb Works

To understand why an integral formula which gives the correct answer for constant functions is usually the correct formula in general, let's go back to the canonical example of velocity and distance.

The main signicance of the fact that the relationship between distance and velocity is cumulative is that if we can break the time interval from t_0 to t_1 up into pieces, and if the formula we are trying to prove is correct on each piece, then the formula must be correct for the whole interval (since the distance corresponding to the whole interval is the sum of the distances on the separate pieces).

Now we know that if $v(t)$ is a constant function, then the distance traveled by an object whose velocity is $v(t)$ will be given by the area under the graph of $v(t)$.
And we also know that the relationship between velocity and distance is cumulative. It follows that the equality between distance and the area under the graph of the velocity function will valid for any function which is made up of a number of pieces where each piece is a constant function. In the trade, a function of this sort is called a **step function.** However if we fill in the area underneath each horizontal piece, what a step function really looks like is a bar graph. The fact that distance corresponds to area in the case of constant functions means that the area in each band of the bar graph corresponds to the distance an object would travel in that little piece of time if its velocity were given by the height of the bar graph (step function) at that point.



An object travels with a constant velocity $v(t) = 2$ cm/sec from time $t = 1$ to $t = 4$. The distance traveled is the same as the area under the graph.

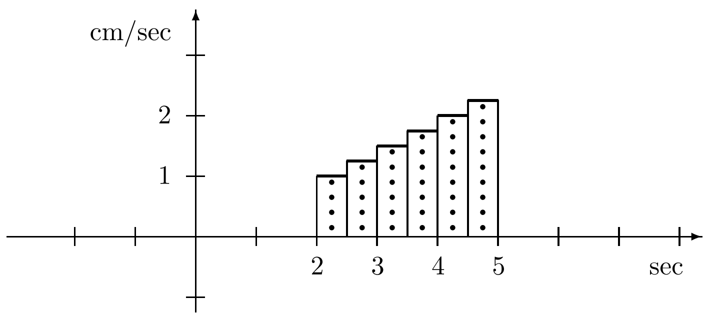
Adding all the pieces together, we see that the area comprised by this bar graph is equal to the distance an object would travel if its velocity were given by the step function (bar graph).

Now let's go back to the question we raised earlier: Suppose that an object is traveling between time $t = 2$ and $t = 5$ and that its velocity at time t is, say, $v(t) = t/2$. How can we "multiply" velocity by elapsed time in a case like this when velocity changes as time progresses?

We claimed above that the answer to this conundrum can be obtained by measuring the area under the graph of the velocity function between the starting time and ending time.

As a way seeing why this is true, we imagine temporarily that instead of increasing continuously, the velocity actually changes by making a very large number of extremely tiny quantum jumps. By making the time interval between jumps small enough, we can get something that approximates that

A Velocity Determined by a Step Function



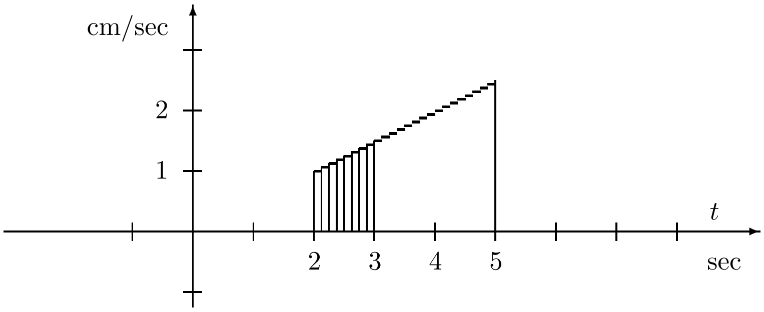
An object travels from time $t = 2$ until $t = 5$ starting at a velocity of 1 cm/sec. Every half second, the velocity increases by .25 cm/sec, and is constant in between jumps. (Thus, for instance, during the final half second, between time $t = 4.5$ and $t = 5$, the object is traveling at a velocity of 2.25 cm/sec. It thus travels a distance of $.5 \times 2.25 = 1.125$ cm during this final half second.) The distance traveled by the object can be easily computed as

$$.5 + .625 + .75 + .8725 + 1.0 + 1.125 = 4.875\,\text{cm}$$

which is numerically the same as the area comprising the bar graph.

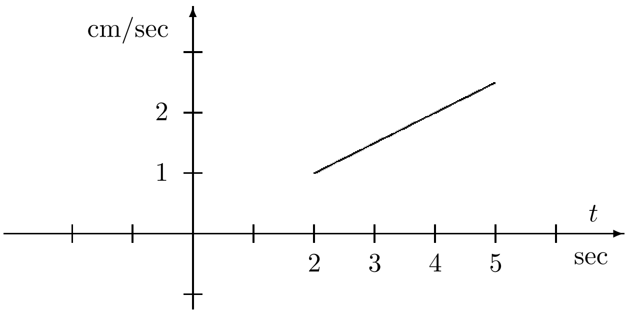
the actual velocity function $v(t)$ (or in fact any velocity function that occurs in physics books and calculus books; any continuous function, for instance) extremely closely.

In other words, the given velocity function can be approximated extremely closely by a step function. For instance, here is a step function that looks fairly close to the curve $v(t) = t/2$ between $t = 2$ and $t = 5$.



A crude step-function approximation to the graph of $v = \frac{1}{2}t$
with a few of the vertical lines of the corresponding bar graph.

In this approximation, the length of the horizontal lines, usually denoted by Δt, is .125. Despite the fact that those with poor vision (especially those with astigmatism) may have difficulty in distinguishing the individual horizontal lines that make up this step function, by mathematical standards, this approximation is quite crude. If we use a Δt which is one-quarter this size, we get the following graph, we is starting to fall within the range where the eye (and the laser printer) are unable to distinguish it from the line $v = t/2$. And yet in terms of the sheer mathematics— setting aside the questions of vision and drawing— we can do much much better.



A step-function approximation to the graph of $v = \frac{1}{2}t$ *with* $\Delta t = .03125$.

If we make Δt any smaller, then we fall below the level of resolution that most laser printers can easily deal with. And yet, at least, in principle, we could make the approximate far better. We could construct a step function where Δt is smaller than the eye can distinguish, or, for that matter, smaller than the diameter of an electron. At that point, for all practical purposes there would be no difference between the step function and the function we started with.

Now at each step, we compute the distance traveled by multiplying the velocity at that step by the width of the step. Adding these all together, we see that the total distance traveled equals the area under the step function. As we make the steps smaller and smaller, we get closer and closer to the distance actually traveled. But we are also getting closer and closer to the area under the graph of the original velocity function. Case closed.

Case closed?

Hm ... At the very least, it would be worthwhile to spell out the reasoning here more carefully. And when we do that, it will turn out that there are a couple of loose ends that need to be tied up. But essentially, except for the fine points, this is the reasoning that shows that the area under the graph of a velocity function equals the distance traveled by an object whose velocity is described by that function. In fact, a lot of students may not want to read any further. But let's look at the fine points.

Passage to the Limit

The fact that we can find the distance traveled by an object whose velocity is described by a step function by measuring the area under the graph of that function, plus the fact that there always exists a step function which approximates a given function to within an accuracy so great that neither the human eye nor electronic microscopes can distinguish the two seems to indicate that the *Distance* = *Area* principle is true to within a very high degree of accuracy.

Let's consider, in fact, the question of exactly how much accuracy we can claim. By taking $\Delta t = 10^{-7}$, we can construct a step function that approximates the function $v(t) = t/2$ (with t ranging from 2 to 5) to within an accuracy better than 6 decimal places. It would seem reasonable to conclude, then, that the *Distance* = *Area* principle is true for the velocity function $v(t) = t/2$ at least to within an accuracy of 6 decimal places. (Perhaps this reasoning is not quite as careful as it ought to be, but it's not off by much. We'll show later how to get a quite precise estimate of the error involved.)

But we could just as well take $\Delta t = 10^{-16}$, and thus achieve an accuracy of 15 decimal places. Or by choosing a step function with still smaller steps, we could achieve an accuracy of 100 decimal places.

If we now consider all possible step function approximations to a given function, we can see that the *Distance* = *Area* principle is true up to any conceivable degree of accuracy. In other words, the principle is just plain true, period.

This reasoning is completely different from what one sees anywhere in pre-calculus mathematics and it is the very heart of what makes calculus different from high school algebra. It goes back to what Archimedes called *the Method of Exhaustion.* Namely, **in calculus one uses the idea that by taking a sequence of closer and closer approximations one can finally arrive at a limit which is exact, even though none of the approximations themselves are exact.**

Stating the reasoning above in more conventional mathematical language: as we consider all possible step functions approximating the velocity function, the area under these step functions converges to the area under the velocity function, and the distance corresponding to these step functions converges to the distances traveled by an object whose velocity is given by the original function. But for the step functions, the area and the distance traveled are the same. Therefore they converge to the same limit, so the area under the original velocity function and the distance traveled by the object are the same.

If we use an arrow to indicate convergence to a given function as we take step functions where the width of the steps become smaller and smaller, we can can graphically show this reasoning by the following diagram:

$$
\begin{tikzcd}
\text{Area of step function} \arrow[r] \arrow[d, equal] & \begin{array}{c}\text{Area under graph}\\ \text{of given function}\end{array} \\
\begin{array}{c}\text{Distance corresponding}\\ \text{to step function}\end{array} \arrow[r] & \begin{array}{c}\text{Distance corresponding}\\ \text{to given velocity function}\end{array}
\end{tikzcd}
$$

The Step-function Approximation Principle

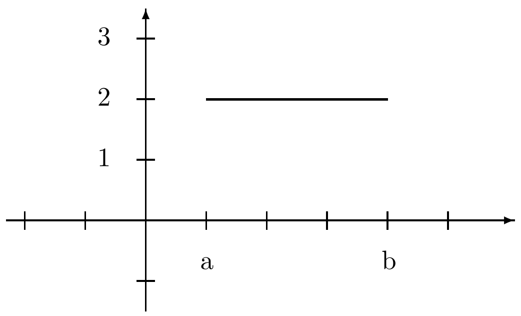
The preceding reasoning, which we have given in terms of the relationship between velocity and distance, applies just as well to the relationship between force and work, between cross-sectional area and volume, and to all the other mathematical relationships we have considered.

In fact, this reasoning seems to completely establish the Second Rule of Thumb: if a formula given by an integral yields the correct answer for constant functions, then it is the correct formula. Unfortunately, however, the reasoning given is flawed, because it depends on a hidden assumption.

There do in fact exist a few cumulative relationships where the Second Rule of Thumb is not valid. The most common of these is the calculation of the length of the graph of a function $y = f(x)$. The formula

$$\text{Length} = \int_a^b dx$$

gives the correct answer for the length of the curve $y = f(x)$ between $x = a$ and $x = b$ in the case when $f(x)$ is a constant function (since in this case Length $= b - a$), and yet is not correct in any other case. (Notice that the function $f(x)$ is not even part of the integral.)



The graph of the constant function $f(x) = 2$. The length of the graph between $x = a$ and $x = b$ is $b - a$.

Step Function Approximations for the Motion of a Falling Object

Stated in practical terms, the fact that the relationship between the velocity of an object and the distance it travels satisfies the Step Function Approximation Principle says that if one only knows the velocity at some finite (but very large) number of time points and computes distance by making the assumption that the velocity in between these time points is constant, then by using enough different time points one can get an arbitrarily good approximation to the true distance the object travels. Let's try this out for the case of a falling object.

The velocity function of a falling object is $v(t) = 32t$ ft/sec, (where t is measured in seconds). We'll see what happens when we approximate this by a step function. To start with, let's assume that we are given the velocity of the object at intervals of .1 second and make the approximating assumption that the velocity is constant in between these time points. Thus, for instance, we might assume that during the first tenth of a second the object's velocity is 0. (This is obviously incorrect, but we are using it as an approximating assumption.) During the next tenth of a second, we take the object's velocity as 3.2 ft/sec, and the corresponding distance is $3.2 \times .1 = .3$ ft. Adding up the corresponding distances, we get a value of

$$0 + .32 + .64 + .96 + .128 + \cdots + 8.96 + 9.28 = 139.2 \text{ ft}$$

for the distance the object falls during 3 seconds. Since the true value is $16\,t^2 = 16 \times 9 = 144$, we see that our approximation is considerably on the low side.

Of course we didn't have to take the lowest possible velocity during each time interval as the value of the step function during that interval. It would also have been reasonable to have assumed that the velocity is 3.2 ft/sec during the first tenth of a second, 6.4 ft/sec during the second one, and 9.6 ft/sec during the third .1 second. This would give an approximation of

$$.32 + .64 + .96 + \cdots + 9.28 + 9.6 = 148.8 \text{ ft}$$

for the total distance traveled, which is as much too large as the original approximation was too small. We can notice, though, that although neither of these two approximations is very good, the two approximations do bracket the true value of the distance the object falls.

To improve the accuracy, we might consider, for instance, a step function where the width of each step is .01 sec. (Thus we would be using 300 time points as the basis for our approximation.) For a low-end approximation, we would assume that the object was traveling at a velocity of 0 during the first hundredth of a second, a velocity of .32 ft/sec during the second 0.1 second, etc. For a high-end approximation, we would assume a velocity of .32 ft/sec for the first .01 sec, .64 ft/sec for the second .01 sec, etc. Without doing the arithmetic, let's note that it is clear that once again the low-end approximation and the high-end one will bracket the true value for the distance fallen. Furthermore, notice that during the final .01 sec, the lower step function uses a value of $2.99 \times 32 = 95.68$ ft/sec and the higher one uses a value of $3 \times 32 = 96$ ft/sec. Thus the discrepancy between the two approximating velocities used during the final tenth of a second is .32 ft/sec. Furthermore, this is the biggest discrepancy for any of the time intervals. Thus we can say that over each time interval, the difference between the higher step function and the lower is at most .32 ft/sec. This means that the difference between the distance computed over the 3 second interval on the basis of the higher step function and that computed on the basis of the lower will be smaller (considerably smaller, in fact) then $3 \times .32 = .96$ ft. Since these two approximations bracket the true value, we can thus conclude that the error in these approximations is smaller than .96 ft.

At this point, we are approaching an accuracy that might be satisfactory for many engineering purposes. But beyond this, we can see from this logic that by taking a time interval of, say, .0001 sec, we would get an error smaller than .0096 ft, and in fact, by using sufficiently short time intervals one could get any desired degree of accuracy.

SIDEBAR: The Mean Value Property

In going through the calculation for a falling body, we defined two step functions. For one of these, we made the value of the step function at a point t between t_i and t_{i+1} equal equal to the value $v(t)$ takes at the beginning of this interval. This choice was obviously too small. For the second step function, we chose the value the $v(t)$ takes at the right end of the interval, which was clearly too large.

It might have occurred to the reader that it would have been more intelligent to have chosen the value that $v(t)$ takes in the middle of the interval. Or perhaps one could choose the average of the values at the two ends.

The question of how to estimate an integral most efficiently by using step functions is essentially the topic of numerical integration and is not really the concern here. It is interesting to notice, though, that in the case of the velocity function $v(t) = 32t$, if one chooses the value that $v(t)$ takes in the middle of each interval, then the answer obtained is exactly correct, regardless of the size of Δt. This is essentially an accident. More precisely, it is true for any *linear* function.

More generally, though, given any reasonable (for instance, continuous) function $v(t)$ defined between points $t = a$ and $t = b$, then for any Δt, even a large one, there exists some step function approximation $v_1(t)$ to $v(t)$ such that the approximation to $\int_a^b v(t)\,dt$ obtained by using $v_1(t)$ will be exactly correct. In other words, if one divides the interval $[a, b]$ up into a sequence of n points $t_0 = a$, t_1, t_2, $\ldots$, t_n, where the distance between each pair of points is some pre-assigned Δt, then it is possible to find numbers C_1, C_2, C_3, etc, such that each C_i lies somewhere in between the smallest and the largest value that $v(t)$ takes on the interval $[t_{i-1}, t_i]$, and so that $\sum_1^n C_i\,\Delta t = \int_a^b v(t)\,dt$. (In fact, if the original velocity function $v(t)$ is continuous, then one can choose $C_i = v(\bar{t}_i)$ for some $\bar{t}_i$ with $t_{i-1} \le \bar{t}_i \le t_i$.)

This is because the relationship between velocity and distance has the **Mean Value Property**. Namely, if an object travels from time t_0 to time t_1 according to a continuous velocity function $v(t)$, then there exists a time $\bar{t}$ in between t_0 and t_1 such that the distance the object travels equals $(t_1 - t_0)\,v(\bar{t}\,)$. Restated, this says that there is some moment in between t_0 and t_1 when the velocity of the object is the same as the average velocity over the whole interval.

The reason for this is easy to see. Consider all the possible values which can be obtained in the form $(t_1 - t_0)v(t)$, where t lies somewhere between t_0 and t_1. Some of these values are clearly less than the actual distance the object travels. For instance, if we choose a time t when the velocity takes its minimum value, then $(t_1 - t_0)v(t)$ gives a value which is too low. On the other hand, for some t, $(t_1 - t_0)v(t)$ is too large (for instance when $v(t)$ takes its maximum value). (If a car travels for an hour at speeds which are always between 30 mph and 50 mph, then the distance traveled will be greater than 30 miles but less than 50 miles. We're assuming here that $v(t)$ is not a constant.) But since $(t_1 - t_0)v(t)$ is a continuous function of t, as it varies between values that are too low and values that are too high, somewhere the must be a time t where the value of $(t_1 - t_0)v(t)$ is exactly correct.

A cumulative relationship between a function $f(x)$ and a quantity Q will always have the Mean Value Property whenever (1) it is an increasing relationship; and (2) $Q = (x_1 - x_0)f$ whenever $f(x)$ is a constant f and is applied between x_0 and x_1.

Furthermore, by being sufficiently devious one can sabotage the Second Rule of Thumb even in cases where it ought to work. For instance, the formula

$$\text{Distance} = \int_{t_0}^{t_1} v(t) + 8v'(t)\,dt$$

gives the correct answer for constant velocity functions, since if $v(t)$ is a constant then the derivative $v'(t)$ is 0, and yet it is wrong in almost all other cases.

For this reason, it's a good idea to understand the hidden assumption underlying our proof of the Second Rule of Thumb, even though when applied to the examples we have been considering this assumption is so natural that most calculus books take it for granted without even mentioning it.

The Hidden Assumption

In the reasoning above we have taken it for granted that if a step function is an extremely good approximation to the actual velocity function describing the motion of an object, then the distance calculated by using this step function will be very close to the actual distance traveled by the object. For convenience, I will call this principle **the Step Function Approximation Principle**. The Step-function Approximation Principle, which applies not just to the relationship between velocity and distance, but also to that between cross-sectional area and volume, force and work, pressure and force, as well as to many other cumulative relationships, is the missing piece we need in order to conclude from the reasoning given previously the a formula given by an integral will be correct provided that it gives the correct answer for constant functions. This principle is in fact valid for most cumulative relationships. Unfortunately, though, there are a few exceptions.

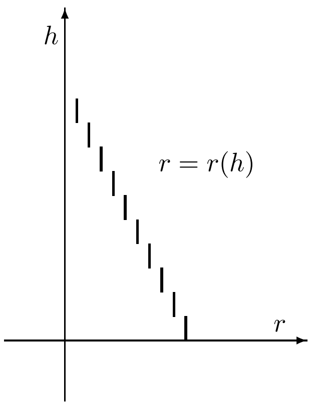
The length of a curve is an example where the Step-function Approximation Principle is not valid. Even if one chooses a step function which is extremely close to a given non-function, the length of the step function will not be close to the length of the given function. Consequently, as seen above, one can't find the correct integral formula for the length of the graph of a function by looking for a formula which gives the correct answer for constant functions

> *The Step-function Approximation Principle is an acid test for formulas given by integrals.* If the quantity in question cannot be approximated to within an arbitrary degree of accuracy by replacing the function in question by step functions, then one cannot find a correct integral formula to express this relationship merely by choosing one which gives the correct answer for constant functions.

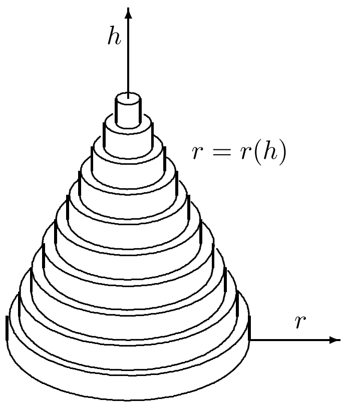
In most calculus books, formulas for an application of integration are developed by first constructing approximations for the quantity in question by using step functions and then taking the limit as the size of the steps goes to zero. This is what one is doing in the "disk method" for finding the volume of a solid of revolution, for instance. To think of a solid of revolution as being

SIDEBAR: The Volume of a Solid of Revolution

We have mentioned before that the volume of a solid can computed as the integral of its horizontal or vertical cross-sections. This can be justified by the Step-function Approximation Principle. Consider in particular the case of a solid whose horizontal cross sections are circles. If the radius $r(h)$ of the cross-section at height h is a step-function, this would say that the solid consists of a stack of disks.



We assume that the horizontal radius of the solid of revolution at height h is determined by a step-function $r(h)$. (Note that the independent variable here is vertical, so that the graph is turned 90° from the expected orientation.)



The solid looks like a stack of disks. Each disk has a cross-sectional area of $\pi r(h)^2$, and thus has a volume $\pi r(h)^2\,\Delta h$. This formula can also be written as

$$\pi \int_{h_1}^{h_2} r(h)^2\,dh,$$

since by assumption $r(h)$ is a constant between h_1 and h_2.

The formula

$$\text{Volume} = \pi \int_0^H r(h)^2\,dh$$

for the volume of a solid of revolution is correct under the assumption that the radius $r(h)$ is a step-function of h, since it is correct for the case of a disk (i. e. cylinder) and the volume of the whole is the sum of the volumes of the disks.

Since it seems intuitively clear that as we make the width of the steps smaller and smaller, the resulting solid can be made to approach any desired solid of revolution arbitrarily closely and that the volumes will also converge to within any desired degree of accuracy, we see that the Step-function Approximation Principle applies and so the formula

$$\text{Volume} = \pi \int_0^H f(h)^2\,dh$$

is valid for any solid of revolution around the vertical axis, where $f(h)$ denotes the radius of the horizontal cross-section at height h.

approximated by a bunch of disks is simply to think of the original function $f(x)$ which was revolved around the x-axis as being replaced by a step function. (See sidebar.)

However this is completely unnecessary. **You don't need to actually use step functions in order to set an integral up.** Step functions are needed for the proof, not the actual calculation. You simply need to find a formula that gives the right answer for constant functions.

What is crucial, though, is to know that one could *in principle* get an arbitrarily good approximation if one did approximate the given function by a step function.

This crucial step, however, is the one that most calculus books give very little attention to. The conventional treatment of applications of integration in most calculus books often assumes without justification that the Step-function Approximation Principle will apply to the particular application under consideration. ("As the thickness of the disks goes approaches 0, the corresponding volume will approach the volume of the given solid of revolution.") For most applications, this is highly plausible. Furthermore, in trying to justify this assumption more rigorously, one runs into the problem that there's the same difficulty in defining concepts such as work, volume, and the like precisely that there is in defining the concept of area rigorously. In fact, in most physics books these concepts are simply *defined* by formulas in the form of integrals. Work, for instance, is defined to be the integral of force with respect to distance.

Stability

What is at issue in deciding whether the Step-function Approximation Principle applies in a given situation is not really about step functions at all. Rather, borrowing a word from some other parts of mathematics (and perhaps not using it quite correctly), the issue is one of *stability.* The relationship between velocity and distance is *stable*, meaning that if one changes the velocity function of an object by a very small amount (or imagines two objects whose velocity functions are very close to each other), then the distance traveled will not be very different. Likewise the relationship between the cross-sectional areas of a solid and its volume is stable: if the solid is changed in such a way that the cross-sectional areas are only slightly different, then the volume will also change very little.

The notion of stability rectifies the flaw in my earlier proof of the Second Rule of Thumb. With this flaw remedied, this becomes no longer a rule of thumb but a theorem.

> Suppose that one is looking for a formula for a variable quantity Q [for instance, work] that is determined by the values of a function $f(x)$ [such as force] for x between $x = a$ and $x = b$. Suppose that the relationship between the function $f(x)$ and the quantity Q is cumulative and is stable in the sense that if one makes a very small change to the function $f(x)$ then the resulting change in Q will also be small. In this case, the formula for Q is given by an integral. Furthermore, if a reasonable integral formula gives the correct result in the case of constant functions, then this formula is in fact correct.

SIDEBAR: "Reasonable" Integral Formulas

The phrase "reasonable integral formula" occurs above because, as an example further on will show, by being sufficiently diabolical, one can indeed contrive exceptions to the principle above: namely formulas which give the correct answers for constant functions and yet fail for other functions. One will have a "reasonable integral formula" if the expression one is integrating is obtained from the basic function in question by applying some continuous function of two variables to it and x.
Assuming that $f(x)$ is the basic function involved, the following are examples of legitimate integrands when applying the Step-function Approximation Principle.

$$\int_a^b f(x)^2 + 4f(x)\,dx$$

$$\int_a^b \frac{x\,dx}{f(x)^2+1}$$

$$\int_a^b x^2 e^{-f(x)}\,dx\,.$$

(As will be indicated below, the most common way to go wrong is to use an integrand that involves $f'(x)$ or $f''(x)$, etc.)

A More Complicated Example: A Volume of Revolution

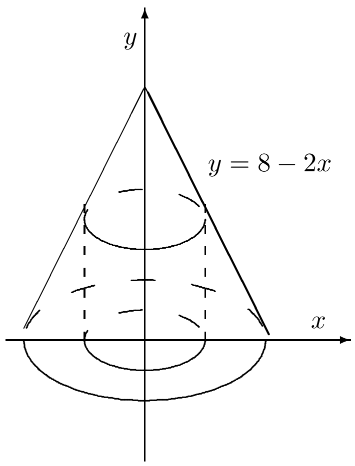
Not all important formulas given by integrals, have the simple form $Q = \int_a^b f(x)\,dx$. As an example of how the principle above can be applied in a more complicated situation, consider the classic problem of determining the *volume of revolution* resulting from revolving a curve $y = f(x)$ **around the y-axis.** (The situation is much simpler if one revolves the curve around the x-axis.)

Now if the function $f(x)$ is a constant H, then the volume of revolution is a cylinder with radius b and height H, and its volume is known to be $\pi b^2 H$. We want to see how to use this to derive the integral formula for the volume when $f(x)$ is not a constant.

For purposes of explanation, instead of merely considering a cylinder, it is essential to consider the volume *between* two concentric cylinders, which looks like a cylinder with a hole.

If the radius of the inside cylinder is a and the outside radius is b, then the volume in between is obtained by simply subtracting the volume of the inside cylinder (the "hole") from that of the cylinder as a whole. This gives

$$\text{Volume} = \pi b^2 H - \pi a^2 H = \pi(b^2 - a^2)H\,.$$



This cone can be seen as the solid resulting from revolving the line $y = 8 - 2x$ around the y-axis. As previously discussed, the volume can be seen as determined by its horizontal cross-sections, whose area at height y is $\pi(y-8)^2/4$, giving a formula

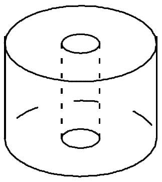
$$\textit{Volume} = \pi \int_0^8 \frac{(y-8)^2}{4}\, dy\,.$$

But the volume can also be seen as determined by cylindrical vertical cross-sections (indicated by the dashed vertical lines), whose area at a distance x from the origin are given by $2\pi x(8-2x)$. This suggests a formula

$$\textit{Volume} = 2\pi \int_0^4 x(8-2x)\; dx\,.$$

It is not easy, though, to see how to verify the correctness of this formula.

We now want to replace this by an integral formula, where the constant height H is replaced by a function $f(x)$.



This is a bit perplexing, though, because it's hard to see what the factor $(b^2 - a^2)$ should become in the integral formula. Simply taking

$$\text{Volume} = \int_a^b \pi f(x)\, dx \qquad (?)$$

is clearly not going to work, because when $f(x) = H$ this gives the incorrect answer $\text{Volume} = \pi b H - \pi a H = \pi(b-a)H$. To try and remedy this by writing

$$\text{Volume} = \pi \int_a^b f(x)\,(dx)^2 \qquad (?)$$

doesn't even give a well-formed integral. The formula

$$\text{Volume} = \pi \int_a^b f(x)\, d(x^2) \qquad (?)$$

seems equally nonsensical. (Actually, this last one can be justified theoretically, and if interpreted in the right way is actually correct. But, for beginners at least, it just looks too flaky.)

To find the correct formula, slightly rewrite the formula for the case $f(x) = H$ (a constant):

$$V = \pi H(b^2 - a^2) = \pi H x^2 \Big|_{x=a}^{b}\,.$$

This way of writing it make it easy to see that if we want an integral formula $V = \int_a^b * * *\, dx$ that will produce this result, we simply need an integrand that will produce $\pi H x^2$ as its anti-derivative when

H is constant. In other words, we need an *anti*-anti-derivative for x^2. But an anti-anti-derivative is simply a derivative, and the derivative of x^2 is $2x$. So to produce the correct answer when $f(x) = H$, a constant, we should integrate $2\pi x H$. Thus

$$V = \int_a^b 2\pi x f(x)\,dx$$

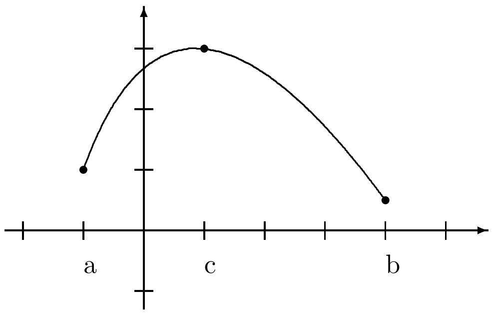
should be the desired formula. In fact in the case, if $f(x) = H$ (a constant), we get

$$\begin{aligned}\text{Volume} &= \int_a^b 2\pi x H\,dx \\ &= \pi H \int_a^b 2x\,dx = \\ &= \pi H(b^2 - a^2)\,,\end{aligned}$$

which is the correct answer. Since the formula yields the correct answer when $f(x)$ is a constant H, it is the correct formula in general.

A Relationship That Is Not Stable

As an example of a mathematical relationship where one does not have stability, and where the Step-function Approximation Principle does not apply, consider the length of the graph of a function $y = f(x)$ between two points $x = a$ and $x = b$. The length of this curve surely has a cumulative relationship to the function, since if c is a value of x between a and b, then the length of the entire curve can be obtained by adding together the length of that portion between a and c plus the length of the portion between c and b. Therefore it is almost certain that the formula for the length will be given by a an integral.



The length of the curve for x between a and b is the sum of the length of that portion between $x = a$ and $x = c$ plus the length between c and b.

However the length of a curve is not stably related to the function determining the curve. One can change the function in such a way that at no point is the change very large, and yet the change in length is enormous. One can, for instance, walk straight down a street in such a way that one's path is a straight line. Or one could walk down the same street, but this time crossing from one side to another every few feet. The new criss-crossing path would never be that far away from the original straight-line path, but the distance one walks would be enormously longer.

(This idea occurs in the theory of fractals. One can start by taking a relatively nice curve, and then change it by adding little bumps all along it. One can then change it still more by adding little bumps along the little bumps, and then add still more bumps to those bumps. Eventually one reaches

SIDEBAR: The Leibnitz Approach

Newton and Leibnitz argued fiercely as to which had the right explanation of calculus, although in truth, neither was completely correct. Leibnitz's way of explaining things seems obviously crazy, and indeed is crazy, as befits a German philosopher whose main claim to fame is not his mathematics but the crazy philosophical idea that the world consists of something called monads. (More precisely, a monad is an entire world in itself, centered around one individual. Every person in the world lives in his own monad. Well, never mind.)

But if you can get past the fact that it's crazy, Leibnitz's way of looking at calculus is actually quite nice and gives reliable results. Furthermore, at least his explanation is consistent with the notation we actually use for integrals.

I'm going to suggest an explanation which is slightly different than Leibnitz's, but is still crazy. Namely, suppose that instead of the interval between $x = a$ and $x = b$ being continuous, it is actually made up of a huge number of extremely small quantum pieces. We write dx for the length of each quantum piece. (It's as if dx is the distance from one atom of the number line to the next. In fact, of course, the mathematical number line, unlike lines in the physical world, does not have atoms.) What $\int_a^b f(x)\,dx$ then means, according to this explanation, is that we let x range over the huge but finite number of points between $x = a$ and $x = b$, and at each of those points we compute $f(x)$ and multiply it by dx. Then we add up all these values for $f(x)dx$.

Despite the way that this explanation is wrong and even crazy, it does produce reliable answers, and in my experience it's the way most people think who actually use calculus as a tool.

When applied to the volume of revolution example, this way of thinking leads us to think of the volume as being made up by gluing together an incredible number of incredibly thin concentric sheets. (Anyone who's ever done papier maché will understand the idea.) But these sheets have width dx — much thinner than a sheet of paper. The total volume of the solid will then be equal to the sum of the incredibly small volumes of all these ultra-thin sheets. Now, at a given distance x from the axis of the solid (i. e. the y-axis), the sheet of paper (as it were) will have a length of $2\pi x$ and a height of $f(x)$, and therefore an area of $2\pi x f(x)$. Since the thickness is dx, the volume of the sheet is $2\pi x f(x)\,dx$. (This seems like a completely valid explanation, but a careful calculation will show that it's not. It doesn't take into account the fact that the sheet is curved and the fact that the top edge of the sheet is beveled to match the slope of the graph of $y = f(x)$. However because of the ultra-thinness of the sheet, the error involved is far smaller than dx; so small that it drops below the quantum level and thus disappears. This is the really crazy part of the explanation.) Leibnitz's symbol $\int$ is actually an elongated S (but don't write it that way, unless you want everyone to know what a total dork you are!) So $\int_a^b 2\pi x f(x)\,dx$. means (according to this crazy explanation) that we add up all these tiny little volumes. This will give the total volume of the solid.

Mathematicians generally don't approve of this explanation because it doesn't make sense, and instead put into calculus books rigorous calculations that are so tedious that very few students are willing to go through them.

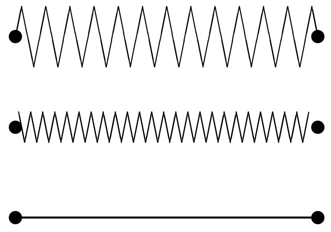
My suggestion is to use the Leibnitz approach to come up with the formula in the first place, and then, if you have any doubts about its correctness, use the principles I've been explaining here: in almost all cases, all you have to do is to check that the formula you came up with gives the correct answer for constant functions.

the points where the changes one is making to the curve become so small that the eye can't even detect them, and yet if one takes this process to the limit one gets a curve which is infinitely long.)

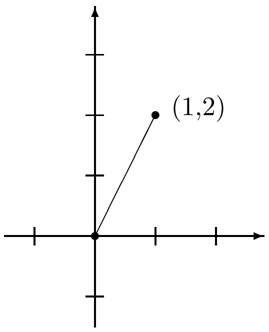
When $f(x)$ is a constant function, then the graph of $f(x)$ is a horizontal line, and its length is simply $b-a$. Thus the formula

$$\text{Length} = \int_a^b dx \qquad (?)$$

gives the correct answer for the length of the graph of a constant function. Nonetheless, this formula does not give the correct result for functions which are not constant. In fact, it always give values which are too small (usually much too small) for functions which are not constant. For instance, if one considers the function $f(x) = 2x$, then its graph is a straight line with a slope of 2, and the length of this line between the points $x=0$ and $x=1$ is easily seen to be $\sqrt{5}$, as contrasted with the value 1 produced by the integral formula above.



A straight-line path and two zig-zag paths between the same two points. The zig-zag paths are $\sqrt{26}$ times as long as the straight-line path. This ratio depends only on the angle of the zig-zags, not on their height. Consequently, we could make the zig-zag path so close to the straight-line path that the eye could not distinguish them, and yet the zig-zag path would be much longer.

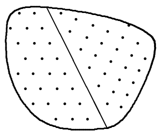


This apparent paradox occurs because one does not get arbitrarily good approximations to the length of a curve by replacing that curve by a step function. In fact, the length of the graph of a step function between points $x=a$ and $x=b$ is always $b-a$. Making the jumps in the step function very small does not affect its length at all. (It is a little strange to even talk about the length of a step function, since the graph has breaks in it. However if we agree that length is cumulative, then it is easy to see that the length of a step function has to be the sum of the lengths of the horizontal pieces. The vertical jumps do not contribute to the length.)

Since telling whether the Step Function Approximation Principle applies can conceivably sometimes be a difficult judgement to make, it's good to have one that's even easier to use.

SIDEBAR: A Double Integral

As an example of a formula given by a double integral, we can consider the relationship between pressure and force. This relationship is cumulative: if a given region is subjected to a pressure given by a function $p(x, y)$ of two variables, and if we split the region into two pieces, then the force on the total region is the sum of the forces on the two separate sub-regions.



Ω

Therefore the relationship between pressure and force will almost certainly be described by a (double) integral. Since $\text{Force} = \text{Pressure} \times \text{Area}$ when pressure is constant, the correct formula in general will be

$$F = \iint_\Omega p(x, y)\, dx\, dy\,,$$

provided that the Step-function Approximation Principle is valid for this relationship.

To get a step-function approximation for $p(x, y)$, we divide the region Ω up into pieces (usually rectangles) and define a function which is constant on each piece. Since the relationship between pressure and force is cumulative, and since the formula $F = \iint_\Omega p(x, y)\, dx\, dy$ is known to be true for constant functions, it follows that it is also valid for this step function. If we make the pieces small enough, then the step function will be a very good approximation to $p(x, y)$: namely, at any point (x, y) of the region Ω, the value of the step function at (x, y) will never be very different from $p(x, y)$.

Since force clearly has an increasing relationship to pressure (making the pressure function larger will always result in a larger force), it follows that the Step-function Approximation Principle applies to this relationship and therefore the formula

$$F = \iint_\Omega p(x, y)\, dx\, dy\,,$$

is valid in general.

Increasing Relationships

The fact that the relationship between velocity and distance is stable and therefore satisfies the Step Function Approximation Principle is common sense, and in this case (as contrasted to what happens in some other parts of calculus) common sense is correct.

However one can give a more rigorous justification for it. To see how this works, let's go back to the example for a falling body between times $t = 0$ and $t = 3$. If the body starts at rest, the velocity function is $v(t) = 32t$ ft/sec. Let's approximate this velocity function by a step function with $\Delta t = .001$.

Now in order to define a step function, we have to make a decision about what value the function takes at each step. In our previous treatment of this example, we saw that two obvious choices were to have make the value of the step function equal to the value of $v(t)$ (i.e. $32t$) at the beginning of the step, and the value at the end of the step. If we call the two corresponding step functions $v_1(t)$ and $v_2(t)$, then the following table gives the general idea. In this table, we let ΔD_1 and ΔD_2 indicate the distances the body would travel during the indicated step if its velocity corresponded to $v_1(t)$ and $v_2(t)$. Thus in the i^{th} row, $\Delta D_1 = v_1(t)\Delta t$ and $D_2(t) = v_2(t)\Delta t$, where t represents any number with $t_{i-1} < t < t_i$. (It doesn't matter precisely what t is chosen, since by assumption the step functions $v_1(t)$ and $v_2(t)$ are constant between t_{i-1} and t_i. For convenience in making the table, we assume that the jump in the two step functions occurs at the *beginning* of each interval. Thus $v_1(t_{i-1}) = 32t_{i-1}$ and $v_2(t_{i-1} = 32t_i$.)

t_i	$v_1(t_i)$	ΔD_1	$v_2(t_i)$	ΔD_2
.000	.000	.000000	.032	.000032
.001	.032	.000032	.064	.000064
.002	.064	.000064	.096	.000096
.003	.096	.000096	.128	.000128
		. . . . .		
		. . . . .		
2.997	95.904	.095904	95.936	.095936
2.998	95.936	.095936	95.968	.095968
2.999	95.968	.095968	96.000	.096000

To find the total distance corresponding to the step functions $v_1(t)$ we need to add the third column of this table, and to find the distance corresponding to $v_2(t)$ we should add the fifth column. It is obviously impractical to do this by hand. However looking at the table closely, one can notice an interesting phenomenon. Namely, the third and fifth columns of the table are almost identical, except for being shifted by one position. Thus when we add these two columns, we get almost the same sums. In fact, it is easy to see that the sums differ by .096000, which is the last entry in the fifth column minus the first entry in the third column, since these are the only entries which do not cancel when we compute $D_2 - D_1$. Thus $D_2 - D_1 = .096$. But as previously mentioned, the true distance traveled by the falling object will lie somewhere in between D_1 and D_2. Thus the discrepancy between the true distance D and the distance as approximated on the basis of either one of the two step functions will be smaller than .096.

SIDEBAR: The Force on a Dam

A standard application of integration treated in most calculus books is the problem of finding the force on a dam, or on the end of an aquarium or tank filled with a liquid.

Here it's important to know the distinction between *force* and *pressure*. Basically, pressure is something that happens at a point, whereas the force resulting from this pressure is something that applies to the entire surface. Pressure is what might conceivably cause the glass in the side of an aquarium (or the face mask of a deep-sea diving suit) to crack. Force is what will cause the end of the aquarium to give way and fall out of its frame.

Pressure is what causes dents in the vinyl tile when a woman wearing stiletto heels stands on it. Force is what causes the floor to cave in when a 900 lb. gorilla stands on it.

When pressure is constant, its relationship to force is given by the equation $\text{Force} = \text{Pressure} \times \text{Area}$. However on the vertical side of a dam or aquarium, pressure is not a constant. The fluid pressure in a liquid at a point is proportional to the depth of the point. More precisely, the formula is $p(x) = \sigma s$, where σ is the density of the fluid. (We assume that the liquid is incompressible, so that σ is a constant.)

The force on the vertical dam surface can be computed as

$$F = \iint_\Omega \sigma x \, dx \, dy \,,$$

where Ω is the region on which the pressure acts. This double integral can quickly be reduced to a single integral. However since this application is usually presented in Calculus I, I'd like to derive it without using the double-integral concept.

We'll assume that the submerged surface on which the pressure acts is not necessarily a rectangle. We let $w(x)$ be the width of this surface at depth x.

As always, the idea is to find a formula that gives the correct answer for constant functions. Taking the width w as constant means that the submerged surface being considered is a rectangle. Now if pressure p is also constant, we have

$$F = p \times \text{Area} = p \times w \times (x_{\text{Bottom}} - x_{\text{Top}})\,.$$

What becomes confusing at this point, though, is the fact that $p(x) = \sigma x$. Thus to make $p(x)$ constant, we should take x to be constant. But since x is the depth of a point on the surface, the only way that x can be constant is to have a rectangle of depth 0, in which case there will be no force.

This is one of those cases which sometimes occur in mathematics where a more general problem is easier to solve than a specific one. If one momentary forgets the fact that $p(x) = \sigma x$, then it is easy to see that the formula

$$\text{Force} = \int_{\text{Bottom}}^{\text{Top}} p(x) w(x) \, dx$$

gives the correct answer when the functions $p(x)$ and $w(x)$ are constant and thus is the correct formula in general. Now we can substitute back $p(x) = \sigma x$ to get

$$\text{Force} = \sigma \int_{\text{Bot}}^{\text{Top}} x w(x) \, dx \,,$$

which is in fact correct.

Now this may not be spectacular accuracy. But by looking at the structure of the table above, one can see if one were to instead use a value $\Delta t = 10^{-12}$, then the approximation for the distance traveled would be accurate to within an error of smaller than 96×10^{-12}. And in fact, by taking Δt small enough, one could achieve any desired degree of accuracy.

This shows that the Step Function Approximation Principle is valid for the velocity function $v(t) = 32t$. But in fact, the reasoning here is easily modified to apply to any (reasonable) velocity function, or function representing force, cross-sectional area, etc. (The calculation above was slightly simplified by the fact that $v(t) = 32t$ is an increasing function. In the general case, one should divide the time interval up into segments on which the velocity function is increasing, and ones on which it is decreasing.)

The only thing used in this reasoning which was really special was the obvious fact that the relationship between velocity and distance is an increasing one, i. e. if one makes a velocity function larger, then the resulting distance will be greater. We used this in order to conclude that if we take step functions $v_1(t)$ and $v_2(t)$ approximating a velocity function $v(t)$, and if $v_1(t) \le v(t) \le v_2(t)$ for all t, and if D_1, D_2 and D denote the corresponding distances, then the true distance D will lie between D_1 and D_2: $D_1 \le D \le D_2$. The fact that D_1 and D_2 can be made arbitrarily close to each other by making Δt small enough can be seen by writing the calculations for D_1 and D_2 in table form. It is then clear that both D_1 and D_2 can be made arbitrarily close to the true distance D.

The reasoning given establishes the following principle:

Increasing Relationships: Suppose that a quantity Q depends on a function $f(x)$ in a cumulative manner, and is an **increasing relationship** — i. e. making the function $f(x)$ larger will always make the quantity Q larger. Then the Step-function Approximation Principle is valid for this relationship, and therefore any integral formula for Q in terms of $f(x)$ which gives the correct answer for constant functions will in fact be valid for all functions.

Unlike the Step-function Approximation Principle, the principle above is not an acid test for finding an integral formula by testing it on constant functions. There do indeed exist relationships that are not increasing but satisfy the Step-function Approximation Principle. However it should be clear that all the examples we started out considering in these notes, namely the relationship between the width of a rectangle and its length, the relationship between velocity and distance, the relationship between force and work, etc. are all increasing relationships. For instance, if two objects travel during the same time period with velocities $v_1(t)$ and $v_2(t)$, and if at every moment t the second velocity is greater than or equal to the first, then clearly the second object will travel further than the first (or equally far in the case when the velocities are always the same). Likewise if two forces $F_1(t)$ and $F_2(t)$ are applied to objects moving between the same two points $x = a$ and $x = b$, and if the second force is always greater than the first, then the work accomplished by the second force will clearly be greater than that accomplished by the first.

Thus Rule of Thumb 2 as well as Rule of Thumb 1 apply to all these situations.

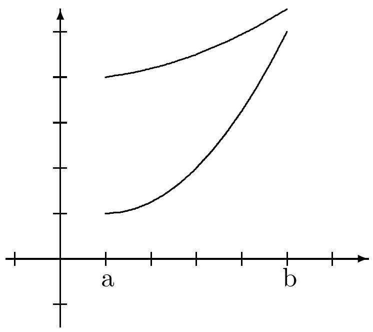
On the other hand, the relationship between a function and the length of its curve is not an increasing relationship. Making a function larger makes its graph higher, but doesn't usually make it longer.

The Length of a Curve

Although the Step Function Approximation Principle does not apply to the relationship betweeen a function $f(x)$ and the length of the graph of that function between two points $x = a$ and $x = b$, on the other hand, length is clearly cumulative, and therefore it is almost certainly given by some sort of integral. Figuring out the correct integral formula, though, is a considerable challenge.

We have noticed that the relationship between a function and the length of its graph is not an increasing relationship, since making the function larger only makes the curve higher, not necessarily longer. On the other hand, what will make the graph of a function longer is to make it steeper.

Given two functions between the same x-coordinates $x = a$ and $x = b$, if one of them is consistently steeper than the other, then that one will be longer. (Here is doesn't matter whether the curve is going up or down, it's only a question of the angle it makes to the horizontal.) This suggests that the slope of the curve, i. e. $|f'(x)|$, might be a key factor in determining its length.



Two curves going from $x = a$ to $x = b$
The steeper curve will be longer.

We can also note that although it doesn't work to try to compute the length of a curve by approximating it by a step function, it *would* seem to make sense to compute it by approximating the curve by a sequence of connected tiny line segments.

This suggests that we might be able to get a correct formula for the length of a curve by finding one that gives the correct result for the special case of a straight line.

If a straight line starts at the point $(a, f(a))$ and ends at $(b, f(b))$, then its length is simply the distance between these two points. According to the Pythagorean Theorem, this would be $\sqrt{(b-a)^2 + (f(b) - f(a))^2}$.

In order to derive a formula that will apply to all curves, we should rewrite this in the form of an integral. From what I've said above, it is plausible that $f'(x)$ should be the key ingredient in this integral.

So we want something like

$$\text{Lenght of Curve} = \int_a^b \sqrt{\ ?\ ?\ f'(x)\ ?}\, dx$$

(where the question marks indicate ingredients that are not yet determined).

It's certainly far from obvious how to do this. But going back to the case of a straight line, if $f(x) = mx + C$, then $f'(x) = m$. To try and relate this to the previous formula

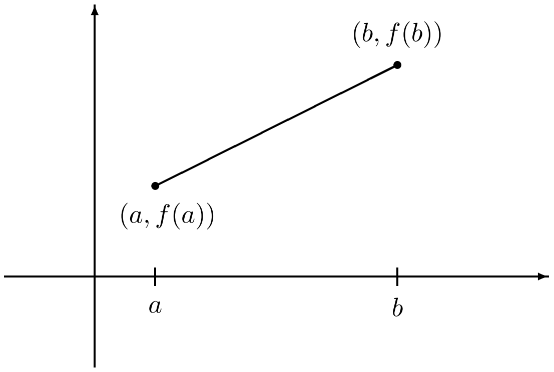
$$\text{Length of Line} = \sqrt{(b-a)^2 + (f(b) - f(a)^2}\,,$$

we could
notice that since m is the slope of the line:

$$m = \frac{f(b) - f(a)}{b - a}\,.$$

We can use this to write

$$\begin{aligned}\text{Length} &= \sqrt{(b-a)^2 + (b-a)^2 m^2} \\ &= \sqrt{(b-a)^2(1+m^2)} \\ &= (b-a)\sqrt{1+m^2} \\ &= \int_a^b \sqrt{1+m^2}\, dx\,.\end{aligned}$$



A straight line $y = f(x) = mx + K$.
The length of the curve will be
$\sqrt{(b-a)^2 + (f(b) - f(a))^2} = (b-a)\sqrt{1+m^2}\,.$

Since for a straight line, $m = f'(x)$, we can rewrite this as a formula

$$\text{Length} = \int_a^b \sqrt{1 + f'(x)^2}\, dx\,.$$

This formula is meaningful for any differentiable function, but is it in fact correct? It turns out that it is, but it's rather hard to justify that rigorously. What we can say is that the formula gives the correct answer for straight lines, and that if we approximate a curve by a sequence of tiny straight lines joined together, then it is extremely plausible that the length of the curve should be very close to the sum of the lengths of these little line segments. When we try and replace "extremely plausible" by a more rigorous argument, we run into the sort of problem we've seen several times before. Namely, we need to know how to rigorously define what we mean by the length of a curve, and it's not obvious how to do this.

So I'll have to be satisfied by saying that the formula given here makes a lot of sense, and in books which do this sort of thing rigorously it is shown to be correct.

(I'll give a slightly more convincing argument a little later.)

Notice that this formula does indeed give the correct answer for step functions — it would have to, since it is correct for *all* functions. The point is, though, that in this particular formula, the step-function case is not decisive. One could change the formula is various ways, for instance

$$\text{Length} = \int_a^b \sqrt{1 + 8f'(x)^2}\, dx \qquad (?)$$

or even

$$\text{Length} = \int_a^b 1 - 4f'(x)\, dx \qquad (?)$$

and these incorrect formulas would still give the correct answer for step functions, since step functions are made up of constant functions and for step functions $f'(x) = 0$. (Of course $f'(x)$ is actually undefined at the jump points of the step function, but this doesn't actually make a difference.)

Derivatives

As mentioned before, the long song and dance given earlier about why distance is the integral of velocity is in some ways a bit silly, since most students will know a much simpler reason for this fact: namely that, almost by definition, velocity if the derivative of distance, and according to the Fundamental Theorem of Calculus differentiation and integration are reverse processes. (This assumes that the function being integrated is continuous.)

> For every integral formula, there is a corresponding derivative formula, and vice versa. If one can figure out one of them, then one also has the other.

This is extremely useful since it's often easier to think about derivatives than integrals.

> The fact that one can use derivatives to figure out integrals seems to be underutilized in most calculus books. The integral is conceptually difficult because it is a "global" concept: $\int_a^b f(x)\, dx$ depends on the entirety of the function $f(x)$ between the points a and b. On the other hand, for given x, $f'(x)$ depends only on the behavior of the function in a very small (in fact, *arbitrarily small*) neighborhood of x and so it's intuitively easier to "see" the derivative $f'(x)$.

On the other hand, in terms of applications, integrals are often more important than derivatives, precisely for the same reason — the derivative only tells how a function is behaving at a single point, whereas the integral shows the cumulative effect of the function over an entire interval.

SIDEBAR: Unreasonable Formulas

The fact that $f'(x) = 0$ when $f(x)$ is a step function can also be used to clear up a point mentioned previously, when I said that when the Step-function Approximation Principle applies to a given situation, even though any *reasonable* integral formula which gives the correct answer for constant functions will in fact be correct, by being sufficiently devious, one can find formulas that fail despite working for constant functions.

Consider, for instance, the canonical example of the relationship between velocity and distance. The Step-function Approximation Principle applies, and therefore we got the simple formula

$$D = \int_{t_0}^{t_1} v(t)\,dt$$

justifying its correctness by the fact that it gives the correct answers when $v(t)$ is a constant function.

However a more outlandish formula also gives the correct result for constant functions, namely

$$D = \int_{t_0}^{t_1} v(t) + 5v'(t)\,dt \qquad (?)$$

Whenever $v(t)$ is constant, $v'(t) = 0$ and so this formula agrees with the previous one for all constant functions, and therefore also for step functions. However for any non-constant function, the result given by this wacko formula is totally wrong.

At first, this seems like a paradox. This wrong formula gives the correct result and the given step functions, and any reasonable function can be approximated arbitrarily closely by step functions, and yet the formula gives the wrong answer for most functions. The reason for this apparent paradox is that the integral here involves $v'(t)$, and v' does not have a stable relationship to v: two functions can be very close to each other in the sense that at no point is the difference between their values very large, and yet their derivatives can be very different.

For instance the function

$$f(x) = \frac{\sin 100,000x}{100}$$

is always between $-.01$ and $.01$, and it is thus so close to the zero function that if one graphed it, the eye could not distinguish the difference. And yet $f(x)$ vibrates back and forth extremely rapidly, so that its derivative is quite large. In fact, since $f'(x) = 1000\cos 100,000x$, one can see that $f'(x)$ oscillates between -1000 and $+1000$.

As earlier stated, when using the Step-function Approximation Principle, the integrand one chooses should not involve the derivative of the function in question (or its second derivative, etc). And more generally, the expression one is integrating should be obtainable by applying some continuous function to the basic function in question.

The principles developed above for setting up integrals can also be used, in a slightly different form, for finding derivatives.

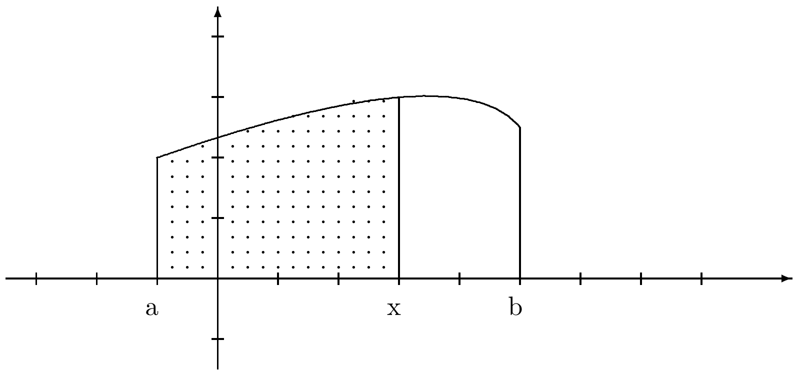
The simplistic high school algebra formulas we started this whole discussion with can rather trivially be re-written as follows:

$$
\begin{aligned}
w &= \frac{\text{Area}}{\text{Length}} = \frac{A}{(x_1 - x_0)} \\
v &= \frac{\text{Distance}}{\text{Time}} = \frac{D}{(t_1 - t_0)} \\
\text{Area of the base} &= \frac{\text{Volume}}{\text{Height}} = \frac{V}{(y_1 - y_0)} \\
\text{Force} = F &= \frac{\text{Work}}{\text{Displacement}} = \frac{W}{(x_1 - x_0)} \\
\text{Pressure} = p &= \frac{\text{Force}}{\text{Area}} = \frac{F}{A} \\
\text{Density} = \rho &= \frac{\text{Mass}}{\text{Volume}} = \frac{M}{V} \,.
\end{aligned}
$$

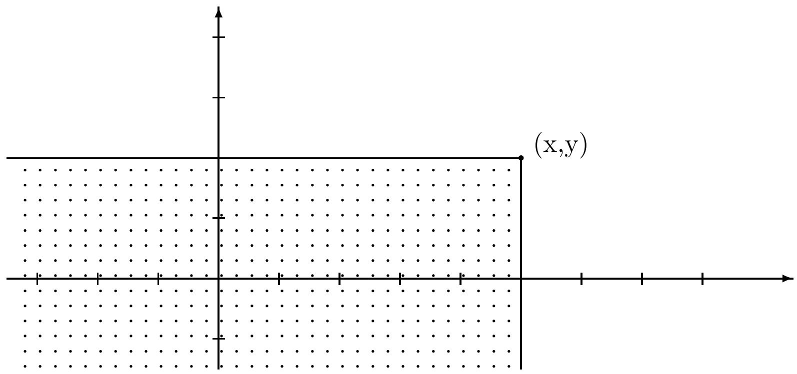
Just as the original six formulas are transformed in calculus into integrals, in calculus the formulas in this version become derivatives:

$$
\begin{aligned}
w(x) &= \frac{dA}{dx} \\
v(t) &= \frac{dD}{dt} \\
A(y) &= \frac{dV}{dy} \\
F(x) &= \frac{dW}{dx} \\
p(x, y) &= \frac{\partial^2 F}{\partial x\, \partial y} \\
\rho(x, y, z) &= \frac{\partial^3 M}{\partial x\, \partial y\, \partial z}
\end{aligned}
$$

Here in the first formula, for example, $A(x)$ represents, as it were, the area of that piece of the plane region *up to* x, i. e. with x-coordinate less than x;



and $F(x, y)$ in the fourth formula represents the total force exerted by the pressure on that part of the region below and to the left of the point (x, y) (as indicated by shading in the figure below).



The principles involved in setting up a derivative are almost exactly the same as those for setting up an integral.

In most cases, a formula for a derivative which gives the correct answer when the derivative function is constant will be correct.

This reasoning is actually analogous to the Step-function Approximation Theorem.

For integrals, we reasoned that a formula which gives the correct answer for constant functions will in fact be correct in most cases, since in most cases one can get arbitrarily close to the right answer if one imagines that the function of concern is made up of extremely small pieces, each of which is a constant function. For derivatives, we don't have to think of the whole function, but only one small piece. If the piece is small enough, then the function might as well be constant, except for an error that can be made arbitrarily small in comparison to the size of the tiny interval. Therefore if we have a differentiation formula which gives the right answer in those cases when the function is constant, that formula will be correct.

Consider again the canonical example

$$\text{Distance} = \text{Velocity} \times \text{Time}.$$

We know that this formula is correct provided that velocity is constant. But also, **it is almost exact if the time interval is very short,** and the relative accuracy gets better and better as we look at shorter and shorter time intervals. (Actually, what is crucial is not that the time interval be short, but that the velocity change very little over this time interval. During a period of rapid acceleration or deceleration, to get much accuracy one needs an extremely short time interval.)

If we now let the variable x represent position, t represent time, and $v(t)$ represent velocity, then the distance traveled between t_0 and t_1 will be $x(t_1) - x(t_0)$, and if $t_1 - t_0$ is short enough, then this will be very nearly equal to $(t_1 - t_0)v(t_0)$.

Now remember that the derivative of x with respect to t at t_1 is defined as

$$\frac{dx}{dt}(t_0) = \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0}.$$

From what was said above, one sees that the numerator of this fraction is almost precisely equal to $(t_1 - t_0)v(t_0)$ and thus

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} \approx \frac{(t_1 - t_0)\, v(t_0)}{t_1 - t_0} = v(t_0),$$

where the approximation becomes arbitrarily good as t gets sufficiently close to t_0. This seems a fairly convincing justification of the well known fact (usually given as a definition) that velocity is the derivative of position with respect to time.

One can usually get an even more convincing argument by using the fact that, in many cases, the relationship of interest is an increasing relationship. In the distance-time example, it is always true that the distance traveled by an object is always greater than or equal to the product of the length of the time interval with the minimum velocity during that time interval, and is always less than or equal to the product of the elapsed time with the maximum velocity. (For instance, if one drives for two hours and the speedometer never drops below 30 m.p.h. and never rises above 50 m.p.h, then the distance traveled will be at least 60 miles, but not more than 100 miles.) It follows from this that whether t_1 is close to t_0 or not, it will be true that the distance traveled by an object between times t_0 and t_1 will be within the following range:

$$(t_1 - t_0)v_{\min} \leq \text{Distance} \leq (t_1 - t_0)v_{\max}.$$

From this, we see that

$$v_{\min} \leq \frac{x(t_1) - x(t_0)}{t_1 - t_0} \leq v_{\max},$$

where $v_{\min}$ and $v_{\max}$ are the minimum and maximum velocity between t_0 and t_1. But as t_1 approaches t_0, $v_{\min}$ and $v_{\max}$ both approach $v(t_0)$ (assuming that velocity is continuous, which in the real world it certainly will be). Therefore

$$\frac{dx}{dt}(t_0) = \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0} = v(t_0).$$

Now it's certainly not news that velocity is the derivative of distance with respect to time. What I like about this way of explaining things, though, is that instead of explaining this by a bunch of plausible hand-waving, it shows that this is an inescapable consequence of basic facts about the relationship between velocity and distance that anyone will be willing to accept as axioms.

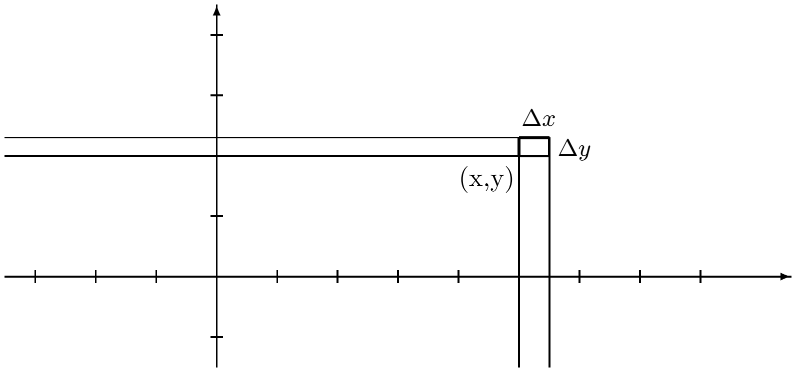
Similar reasoning shows that any time we are considering a possible derivative formula between two variables (distance and time, in this case) and the relationship between the variable on top (distance, in this example) and the function on the left hand side (here, velocity) respects inequalities, then if the proposed formula is correct in the case when the left-hand function is constant, then it is correct in general.

For instance, let $V(h)$ be the volume of that portion of a solid up to height h and let $A(h)$ be the area of the cross section at height h. We have previously asserted that

$$A = \frac{dV}{dh}.$$

In justification of this, we can note first that V has an increasing relationship to A: if we make the cross sections of the solid larger, then the volume will also increase. We can also note that the formula $A = \dfrac{dV}{dh}$ is true when the cross section A is constant, for in that case $V = Ah$. It then follows that the derivative formula is valid in general.

When applied to functions of several variables, it's important to remember that the derivatives which are in reverse correspondance to integrals are the *mixed* partial derivatives. For instance, if $F(x,y)$ denotes the force exerted by a fluid on that portion of a region below and to the left of a point (x,y), then $\dfrac{\partial^2 F}{\partial x\,\partial y}$ will be the limit of the ratio between the force exerted on an extremely small piece of the region at (x,y) and the area of that little piece.



This is a consequence of the formula, for a reasonably nice function $F(x,y)$,

$$\frac{\partial^2 F}{\partial x\,\partial y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{F(x+\Delta x,\, y+\Delta y) - F(x+\Delta x,\, y) - F(x,\, y+\Delta y) + F(x,y)}{\Delta x\,\Delta y}\,.$$

(A little thought shows that the numerator of this fraction equals the force on the tiny rectangle in the diagram whose sides have length Δx and Δy.)

If one now uses the fact that force has an increasing relationship to pressure (increasing the pressure will create a greater force), and that force equals pressure times area when pressure is constant, it then follows that the force on the square whose corners are (x,y), $(x+\Delta x,\, y)$, $(x+\Delta x,\, y+\Delta y)$, $(x,\, y+\Delta y)$ lies somewhere between the product of the area of that square (i. e. $\Delta x\,\Delta y$) with the minimum pressure on the square and the product of the area with the maximum pressure:

$$\text{Minimum Pressure in Square} \leq \frac{\text{Total Force on Square}}{\Delta x \Delta y} \leq \text{Maximum Pressure in Square}\,.$$

(Restated in words: the average pressure over the square lies somewhere between the minimum value that the pressure takes within that square and the maximum value within the square.) It then follows that as one lets the square approach a single point, the force-area ratio approaches the pressure at that point as a limit.

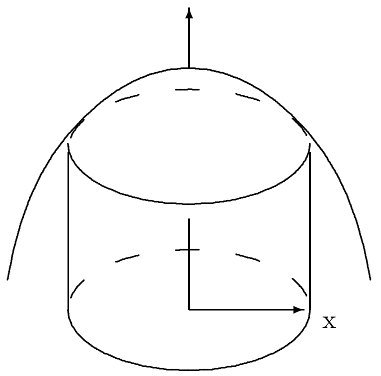
The noteworthy point here is that it's not any more difficult to think this sort of thing through when the dependent variable is two or three-dimensional than it is in the one-dimensional case.

> The example of density and mass is often taken as a prototypical example for the concept of the derivative. Speaking strictly metaphorically, one might say that velocity is the density of distance with respect to time, pressure is the density of force with respect to area, and cross-sectional area is the density of volume with respect to height.

Volume of Revolution (Redux)

To see how this works in a more complication situation, let's return to the problem of revolving a curve $y = f(x)$ **around the y-axis** and see how much simpler it can be to think in terms of derivatives.

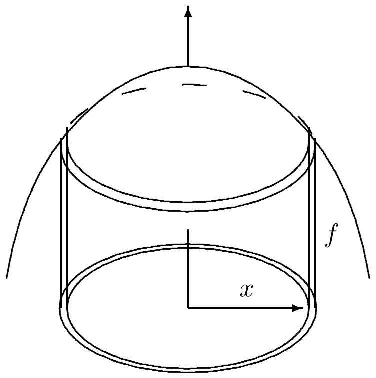
In order to find a derivative formula, one has to think of the volume V as being a function of x. Namely, for fixed a, $V(x)$ will denote the volume obtained by revolving that portion of the curve between a and x around the y-axis.



Now $\dfrac{dV}{dx}$ represents the rate at which the volume is increasing with respect to x. From a naive point of view, it should seem clear that the rate of increase in volume should be equal to the vertical surface area that bounds that part of the solid of revolution which extends out to radius x, for when x increases slightly, the new volume will be a layer along this surface very much like a coat of paint. This surface area is clearly the product of its circumference with its height, namely $2\pi x\, f(x)$. (The radius of this cylindrical surface is x and the height is $f(x)$.)

If this is not obvious, think about it this way: The derivative dV/dx is obtained by considering the amount that the volume increases when when increases x by a tiny amount Δx, and taking the ratio between that increase in volume (ΔV) and Δx. (To get an exact value, of course — which we definitely want — one then has to take the limit as Δx approaches 0. In principle, we should also remember that x can decrease as well as increase, but it's really adequate to think about Δx being positive.) Now if one thinks of Δx as being the thickness of a sheet of paper, then what one is doing is essentially increasing the volume V by wrapping a sheet of paper around it. To fit, the area of this sheet of paper needs to be the area of the exterior vertical surface, name $2\pi x\, f(x)$. Since its thickness is Δx, the volume of the sheet of paper (i. e. the amount of increase in V) is $2\pi x f(x)\Delta x$. Therefore the ratio is $2\pi x f(x)$, as claimed.

This formula, to wit $\dfrac{dV}{dx} = 2\pi x f(x)$, is in fact correct.



The area of the vertical boundary surface of the solid is $2\pi x\, f$.
Consequently $\Delta V \approx 2\pi x\, f\, \Delta x$

Since differentiation and integration are reverse processes, we immediately get the correct integral formula

$$V = 2\pi \int_a^b x f(x)\, dx$$

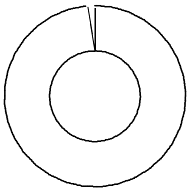
with a great deal less trouble, and, in my opinion, a lot more assurance than before.

The problem comes when some mathematician comes along and complains that the above formula, $\Delta V = 2\pi x f(x) \Delta x$ is not quite correct, convincing though the explanation may have seemed. For one thing, the top surface of the volume of revolution is usually not horizontal, and the means that the edge of the piece of paper in my explanation should be beveled to match the slope of the graph of the function $f(x)$. Now since the paper is only maybe .01 inch thick, beveling the edge is going to make only a minuscule change in the volume of the paper.

But even if the top of the volume of revolution were horizontal— i. e. the function $f(x)$ were constant— the formula given for ΔV would be slightly wrong. In fact, if f is constant, we can do the calculation exactly, since ΔV is the volume between two cylinders, the inside one having a radius of x and a volume of $\pi x^2 f$, and the outside one (if we assume that Δx is positive) having a radius of $x + \Delta x$, and consequently a volume of $\pi(x + \Delta x)^2 f$. Thus the exact formula for ΔV, under the assumption that f is constant, is

$$\begin{aligned} \Delta V &= \pi(x + \Delta x)^2 f - \pi x^2 f \\ &= \pi x^2 f + 2\pi x f \Delta x + \pi f\, (\Delta x)^2 \\ &= 2\pi x f \Delta x + \pi f\, (\Delta x)^2\,. \end{aligned}$$

The discrepancy between this and the original piece-of-paper explanation is $\pi f\, (\Delta x)^2$. If one considers that Δx is the thickness of a sheet of paper, maybe .01 inch, then if the sheet of paper is 10 inches across, $\pi f\, (\Delta x)^2 = 10\pi(.01)^2 \approx .00314$. It's not too surprising that we missed this before.



Still, the sheet-of-paper explanation seemed extremely convincing, and one might wonder what the flaw was. The flaw consists, in fact, of the fact that when one bends a sheet of paper into a cylinder, the two edges don't actually meet perfectly where they join, but leave a little empty wedge. Furthermore, bending the sheet of paper will involved slightly compressing the inner surface and stretching the outer surface. This actually becomes quite perceptible if instead of using a sheet of paper one uses a sheet of rubber an inch or two thick. For the sheet of paper, though, the discrepancy is so tiny that even if we zoom in on it very closely, it will get lost in comparison to the microscopic imperfections inherent in the paper itself.

The mathematician can take his exact formula

$$\Delta V = 2\pi x f \Delta x + \pi f\, (\Delta x)^2$$

and correct derive the same formula I did:

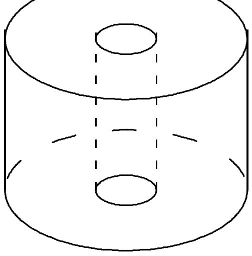
$$\begin{aligned}\frac{dV}{dx} &= \lim_{\Delta x \to 0} \frac{2\pi x f \Delta x + \pi f\,(\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2\pi x f + f\,\Delta x) \\ &= 2\pi x f\,.\end{aligned}$$

All this, however, is still under the assumption that the function $f(x)$ is constant. To deal with the most general case, one must start using inequalities instead of equations. It's certainly quite feasible, but it's not pretty.

I hope that most students are suitably *unimpressed* by the mathematician's rigorous derivation. Not only does it involve a whole lot of work, but it loses the intuitive naturalness of the informal sheet-of-paper explanation. There is something extremely inelegant about going through so much work to get the formula for ΔV exactly correct, only so that one can discard the extra bit when one passes to the limit. The fact is, in my opinion at least, unless one is writing a text-book or mathematical paper, one should never have to work through any equations at all to derive this sort of formula. One should be able to see it at first glance.

In case one's intuition is not that good, though, or one does not completely trust it, one can simply use the principles developed above, namely:

> In order to figure out dV/dx at any specific point x, it is okay to assume that $f(x)$ is a constant function.



When $f(x)$ is constant, we don't need to work out ΔV. We can simply compute dV/dx by using the calculus we already know. If $f(x)$ is a constant f, then the volume of revolution has a horizontal top surface and simply becomes a cylinder with height f and with a hole in it. Thus $V = \pi x^2 f - H$, where H is the volume of the hole. H doesn't change when x changes, so $dH/dx = 0$ and

$$\frac{dV}{dx} = \frac{d}{dx}(\pi x^2 f - H) = 2\pi x f\,.$$

End of calculation!

The fact that we can treat $f(x)$ as a constant when computing dV/dx has to do with the fact that in computing the derivative of the volume at a specific x, we don't care about the volume in its entirety. All we care about is an extremely tiny piece of the volume around x, and once we zero in on a small enough neighborhood of x, $f(x)$ changes very little. This idea is just a variation on the Step-function Approximation Principle. The Step-function Approximation Principle, as applied to this example, says that it really doesn't make much difference to the value of V *or to the rate at which it changes* if we suppose that instead of the curve $y = f(x)$ increasing or decreasing continuously, it actually consists of an incredible number of tiny horizontal pieces. But in computing dV/dx, we're only looking at a very tiny piece of the curve, so it really won't make very much difference if we assume that this piece is horizontal. (You have to think about this for a while. The rate at which V changes does not depend on the slope of the graph of $y = f(x)$ at the specific point x; it only depends on the height of the graph there, i. e. on the value of $f(x)$.) And although there is always *some* error in doing this, albeit quite small, the discrepancy tends to zero when we take the limit as Δx approaches 0.

Curve Length — Redux

If the Step-function Approximation Principle does not apply to a certain relationship, then one cannot correctly compute a derivative by making the assumption that the function which determines that quantity in question is constant.

Let's return to the case of the length of the graph of a function. I want to forget the work that was done previously in developing the formula for length and start over again from scratch using an approach that will turn out to be simpler.

I have mentioned that it's difficult (although far from impossible) to mathematically define the concept of length for a curve rigorously. And yet in the real world we do measure lengths of curved roads both *in vivo,* as it were, and on maps. In fact, in fantasy one could imagine measuring the length of a curve by getting onto it with a tiny automobile and measuring the distance with the odometer.

Now if we let $s(x)$ be the length of the curve $y = f(x)$ as measured starting at a up to x, then $s(a) = 0$ and s is the integral of its derivative:

$$\text{Length of Curve} = s(b) = s(b) - s(a) = \int_a^b \frac{ds}{dx}\, dx\,.$$

Now if we let t represent time, then

$$\frac{ds}{dx} = \frac{ds/dt}{dx/dt}\,.$$

In other words, ds/dx is the ratio between the speed at which the point $(x,\, f(x))$ is moving along the curve and the speed at which the x-coordinate is changing. To simplify, if we let the point move along

SIDEBAR: Distance and Velocity on a Curve

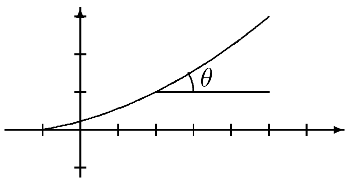
The fact that "length" and "distance" are almost the same concept brings up an interesting point. In a beginning calculus or physics class, distance, as measured along a straight line, is taken as a primitive concept. It is something measured with yardsticks or metersticks or tape measures, and requires no mathematical explanation. In beginning calculus, the idea is that the thing which is difficult to measure simply is velocity, and the question of how to define instantaneous velocity leads to the notion of the derivative.

But on curves, distance is difficult to measure, and, as we shall see, we can actually use velocity as a way to develop a formula for distance.

The reason that velocity is simpler to address than distance on a curve is that distance is a concept which relates to the curve as a whole, and so the non-straightness of the curve becomes a big problem. The velocity at a given instant, on the other hand, is something that is determined by a very small piece of the curve — arbitrarily small, in fact. But if we zoom in on a sufficiently small piece of a differentiable curve, then the curvature is so slight that it's not perceptible to us, any more than the curvature of the earth is perceptible to us human beings standing on a very small proportion of it, and so the curve looks like a straight line.

the curve in such a way that x increases at a constant speed of 1 (i. e. $x = t$), then $\dfrac{ds}{dx}$ is the speed at which the point is moving along the curve.

Now remember that we only need to look at an extremely small piece of the curve, and when we zoom in on this small piece, the curvature is so slight that the curve looks like a straight line.



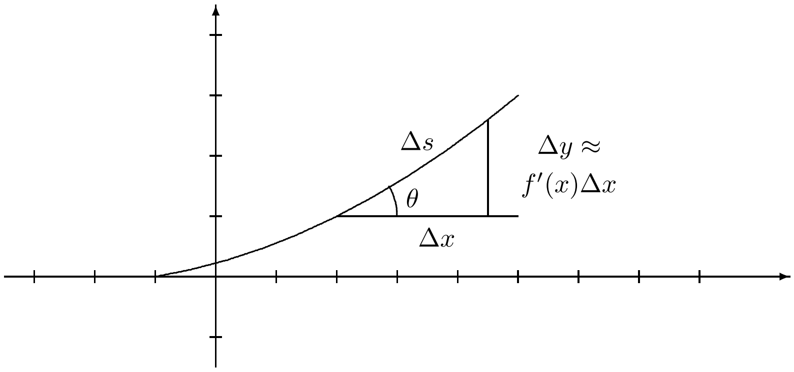
From this, it is easy to see that the ratio between ds/dt and dx/dt is completely determined by the angle that the curve makes to the horizontal. Call this angle θ. In fact, it seems quite apparent that the ratio between the horizontal speed and the slanted speed will be $\cos\theta$. (Suppose that the point moves a short distance Δs on the curve. Then the distance that x moves will be the horizontal side of a triangle, as shown below, so that $\Delta x = \Delta s\, \cos\theta$.)

On the other hand, this same triangle also determines the slope of the curve.

$$\text{Slope} = f'(x) = \tan\theta\,.$$

Now

$$\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+f'(x)^2}\,.$$



Putting all this together (and remembering that ds/dx is positive), we get

$$\frac{ds}{dx} = \frac{1}{\cos\theta} = \sqrt{1 + f'(x)^2}\,,$$

and therefore

$$\text{Length of graph} = \int_a^b \frac{ds}{dx}\,dx = \int_a^b \sqrt{1 + f'(x)^2}\;dx\,.$$

It is reassuring that this answer agrees with the one we saw earlier. I believe that this way of deriving the formula is conceptually simpler than the previous explanation and also more convincing.