Name: ---------------------------------

Question 1

Suppose that V is a complex (i.e. $\mathbb{F} = \mathbb{C}$) inner-product space. Prove that if $N \in \mathcal{L}(V)$ is normal and nilpotent, then $N = 0$.

Question 2

Give an example of an operator on $\mathbb{C}^3$ whose minimal polynomial equals z^2.

Question 3

Let n be a positive integer. Let $A \in \mathbb{C}^{n,n}$ be a square n-by-n matrix (cf. 3.39 in the book for the notation). Clearly, for any integer m,

$$\dim \operatorname{span}(A, A^2, \ldots, A^m) \leq n^2 = \dim \mathbb{C}^{n,n}$$

Prove that in fact a stronger inequality

$$\dim \operatorname{span}(A, A^2, \ldots, A^m) \leq n$$

takes place for any $m > 0$.

Question 4

For a complex number $a \in \mathbb{C}$, let $T_a \in \mathcal{L}(\mathbb{C}^4)$ be an operator whose matrix (with respect to some basis) is

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & a & 2 \end{bmatrix}$$

a) Find the characteristic polynomial of T_a.

Answer:

b) Find the minimal polynomial of T_a

Answer:

c) Is it true that $T_0 = T_1$ (with possibly different choices of the basis)? Prove your answer.

Question 5

Assume that two operators S and T acting on a finite-dimensional vector space V over $\mathbb{C}$ commute:

$$ST = TS.$$

Prove that they must have a common eigenvector.

Hint. Explain why $E(\lambda, T) \neq \{0\}$ for some $\lambda \in \mathbb{C}$, then prove that this space is invariant with respect to S.

Question 6

Let A and B be two n-by-n matrices (with $n > 0$), and let

$$p_A(z) = \det(zI - A) \quad \text{and} \quad p_B(z) = \det(zI - B)$$

Prove or give a counterexample for the following statement:

$$\det(p_A(B)) = 0 \quad \text{if an only if} \quad \det(p_B(A)) = 0$$