The Math of Rational Choice - Math 100 Spring 2015

Mathematics can be used to understand many aspects of decision-making in everyday life, such as:

1. Voting
   (a) Choosing a restaurant
   (b) Electing a leader
2. Apportionment
   (a) Allocate M commodities to N parties
   (b) Example: Number of congressional delegates per district
3. Fair division
   (a) How to cut a cake fairly
   (b) Dividing pirate booty
   (c) Choosing teams

1 Voting

Arrow's Theorem: All voting systems are crap.

(Also called: "Arrow's Impossibility Theorem", the "Dictator Theorem", "Arrow's Paradox")

Arrow, Kenneth J. *A difficulty in the concept of social welfare.* The Journal of Political Economy (1950): 328-346.

Arrow's Theorem: Suppose the members of a finite population each rank some finite number of choices a,b,c,.... There is no way to determine a global ranking of these choices while satisfying all of the following *fairness criteria*:

Universality: For any set of individual voter preferences, the system should yield a unique and complete ranking of the choices.

Independence of irrelevant alternatives (IIA): The outcome shouldn't change if a non-winning candidate is added or removed (assuming votes regarding the other choices are unchanged)

Monotonicity: If one or more votor upranks a choice, it shouldn't *lower* the option in the final tally.

Citizen sovereignty: Every possible final preference order should be achievable by some set of individual preference orders.

Arrow's Theorem (alternate): Suppose the members of a finite population each rank some finite number of choices a,b,c,. . . . There is no way to determine a global ranking of these choices while satisfying all of the following *fairness criteria*:

No dictator: The final ranking shouldn't be determined by one person's preferences (where the person is chosen beforehand)

Universality: For any set of individual voter preferences, the system should yield a unique and complete ranking of the choices.

Independence of irrelevant alternatives (IIA): The outcome shouldn't change if a non-winning candidate is added or removed (assuming votes regarding the other choices are unchanged).

Unanimity: If every individual prefers a certain option to another, then so must the final choice.

Arrow's Theorem (our text): Suppose the members of a finite population each rank some finite number of choices a,b,c,.... There is no way to determine a global ranking of these choices while satisfying all of the following *fairness criteria*:

Independence of irrelevant alternatives (IIA) The outcome shouldn't change if a non-winning candidate is added or removed (assuming votes regarding the other choices are unchanged)

Majority If a candidate is ranked as the favorite by a majority of voters, the final ranking must also rank the choice #1

Monotonicity If one or more votor upranks a choice, it shouldn't *lower* the option in the final tally.

Condorcet If a choice that beats every other choice in pairwise comparisons, the final ranking must also rank the choice #1

Components of a Voting System:

1. Balloting: One choice, several choices ("top n candidates"), ranked choices (Preference ballot), etc.
2. Tabulating: Amalgamating the final ballots into a single decision
3. Outcome: Winner only, Partial ranking, Full ranking

Example: Consider a vote on favorite dessert, in a group of 37 people:

A) Apple Pie

B) Biscotti

C) Cake

D) Doughnuts

The ballots as collected as follows:

14	ABCD
10	CBDA
8	BDCA
4	DCBA
1	CDBA

(By the way, there are only 5 possible rankings here. How many possible rankings were there in theory?)

2 Plurality voting

Often called: "First past the post"

Brief description: Each voter votes for one candidate, and the candidates are ranked by total number of votes.

Variants: If the voter gets as many votes as the number of positions available, this is *Block Voting*. If each voter gets a fixed number between 1 and the number of positions, this is *Partial Block Voting* (or *Limited Voting*).

Factoids : Mathematician Donald Saari calls plurality voting "the only procedure that will elect someone who's despised by almost two-thirds of the voters."

Some 'Pros': 1. Simple and transparent

2. Required by *Roberts Rules of Order*
3. Limited/partial block voting permits limited proportional representation

Some 'Cons': 1. Susceptible to ties, especially if slate is large compared to number of voters

2. Violates the *Condorcet criterion* (see below)

3. Very susceptible to tactical voting
4. Interest blocks are punished for running many candidates (common problem in city council and school board elections)
5. Small cohesive blocks of voters can overpower large disorganized blocks

Dessert Example: A:14, C:11, D:8, B:4; Apple Pie wins.

Notes:

- Apple pie did *not* get a majority. With many candidates, the percentage of the vote needed to win under plurality can be ridiculously low
- Apple pie was *last choice* for the majority!
- Biscotti had 28 1st or 2nd-place votes, seems an obviously better choice
- Suppose we had head-to-head matchings:
  Biscotti beats Apple 23-14
  Biscotti beats Cake 22-15
  Biscotti beats Doughnuts 32-5
  ...so Plurality Voting fails the Condorcet Criterion

Strategic Voting: Instead of "wasting your vote the candidate you really like (but has no real chance of winning) you cast it for someone you like less who has might actually win. Sometimes strategic voting can be used for very strange manipulations.

Suppose the 4 doughnut lovers see they haven't got a chance, but they prefer cake to either biscotti or apple pie. They change their votes from DCBA to CDBA. Now the table looks like:

14	ABCD
10	CBDA
8	BDCA
5	CDBA

Cake is now the top vote-getter (C:15, A:14, B:8)

3 Borda count

Brief description: Every voter ranks candidates in preferential order. Candidates are assigned points depending on the rank (for example, if candidates A, B, and C are all the candidates ranked in that order, A gets 3 points, B 2 points, and C 1 point). A candidate's total score is the sum of the votors rankings. In an election for k choices, the candidates with the top k total scores are chosen.

Factoids: Named for Jean-Charles de Borda (1733 – 1799). Used by the French Academy of Sciences until Napolean took over and imposed his own voting system. A variant was used by the Roman senate.

Variants: By adjusting the way points are allocated, varaints can make it easy to handle partial rankings (where a voter need not rank all candidates) equitably.

Some 'Pros': 1. simple and transparent

2. Is stable to small changes in rankings (not "quasi-chaotic")
3. Relatively immune to strategic voting
4. Favored by some top voting procedure experts in Mathematics
5. Widely used in academic settings for student and faculty legislatures.
6. Gives more weight to a high choice for a large majority than the top choice for a smaller majority or minority, and downweights a candidate strongly opposed by a minority.

Some 'Cons': 1. *Majority paradox:* A candidate can be the top choice of the majority of votors yet not win.

2. Fails the *Condorcet Criterion*

3. A coalition can increase the probability of one of its members being elected by having more of its members on the ballot. (This is the opposite of plurality voting.)

4. *Irrelevant alternative paradox:* Removing a losing candidate from the ballot and recounting can change the outcome.

Dessert example:

14	ABCD
10	CBDA
8	BDCA
4	DCBA
1	CDBA

There are 4 choices, so each choice gets 1-4 points per voter.

A: $14 * 4 + (10 + 8 + 4 + 1) * 1 = 56 + 23 = 79$ points

B: $8*4+(14+10)*3+(4+1)*2 = 32+72+10 = 114$ points

C: $(10+1)*4+4*3+(14+8)*2 = 44+12+44 = 100$ points

D: $4*4+(8+1)*3+10*2+14*1 = 16+27+20+14 = 77$ points

...so Biscotti wins!

4 Digression

Some facts about Condorcet:

Full name: Marie-Jean-Antoine-Nicolas de Caritat, marquis de Condorcet

Born: September 17, 1743

1774: Appointed Inspector General of the Paris mint

Outspoken opponent of the slave trade and the death penalty

1785: *Essai sur lapplication de lanalyse la probabilit des dcisions rendues la pluralit des voix* (Essay on the Application of Analysis to the Probability of Majority Decisions): First to show that voting can become intransitive (Condorcet's Paradox)

1791: Became secretary of the Legislative Assembly of the French Revolution, presided during drafting of new constitution.

1793: Criticized the final result, branded a traitor, had to hide for 8 months

1794? *Esquisse d'un tableau historique des progrs de l'esprit humain* (Outlines of an historical view of the progress of the human spirit)

> "will arrive the moment in which the sun will observe in its course free nations only, acknowledging no other master than their reason; in which tyrants and slaves, priests and their stupid or hypocritical instruments, will no longer exist but in history."

1794: Fled his hideout, was arrested, died two days later in jail

5 Condorcet methods (pairwise comparisons, Kemeny-Young/Votefair

Brief description: A "Condorcet method" is any method that satisfies the *Condorcet Criterion*: If some candidate A is preferred by a majority of voters to every other candidate (possibly a different majority for each alternative) then A should be elected.

One simple Condorcet method is "pairwise comparison" (from the text).

Another currently popular is One such that works well for multiple candidate elections is the Votefair system aka Kemeny-Young, invented by John Kemeny.

Most take the form: every voter ranks the candidates, and every possible ordering of candidates is given a score based on all the 2-way matchups in the sequence. The ordering with the highest score is chosen

Factoids: Named for the marquis de Condorcet. John Kemeny is a Dartmouth mathematician best known as the inventor of the programming language BASIC.

Some 'Pros': 1. Works just as well when a voter does not want to rank all the candidates

2. If every voter prefers A to B, then A is ranked over B in the final ranking (none of the other methods we're discussing satisfy this)

3. Low probability of ties (in Kemeny-Young)

Some 'Cons': 1. Difficult to understand, and computationally difficult

2. Susceptible to *Participation paradox:* adding ballots that rank A over B can increase B's final popularity over A

3. *Irrelevant alternative paradox:* Removing a losing candidate from the ballot and recounting can change the outcome.

4. High probability of ties (in pairwise comparison)

Pairwise comparison: All the candidates are matched head-to-head against one another.

For each pair of candidates A and B, the number of voters preferring Candidate A over B (and vice versa) is tabulated. The candidate receiving more votes in this head-to-head is awarded one point.

If the two candidates split the votes (tie), each gets a half point.

After all pairs have been compared, the candidate with the most number of points wins the election.

Note that if there are N candidates, then there are $1+2+3+\cdots+(N-1) = \frac{N(N-1)}{2}$ comparisons to check.

Dessert example:

14	ABCD
10	CBDA
8	BDCA
4	DCBA
1	CDBA

There are 4 choices, $(4 \times 3)/2 = 6$ comparisons:

1. AB: 14 prefer A to B, B wins
2. AC: 14 prefer A to C, C wins
3. AD: 14 prefer A to D, D wins
4. BC: 14+8 $=$ 22 prefer B to C, B wins
5. BD: 14+10+8 $=$ 32 prefer B to D, B wins
6. CD: 14+10+1 $=$ 25 prefer C to D, C wins

Final scores: A:0, B:3, C:2, D:1 So the ranking is $B > C > D > A$ (Biscotti wins again)

6 Runoff Methods (including Single Transferable Vote/Hare)

Traditional runoff: There is an election; if the plurality candidate does not have a majority, 2 or more of the "top" candidates (determined somehow) run again

Single Transferable Vote: In the initial election ballots allow for the ranking of several or all candidates, and if a "runoff" is necessary it is done automatically. Often called *Single Transferable Vote* (STV) method.

Brief description of STV: Voters rank candidates, and their highest-ranked candidates compete using a plurality vote.
Any candidates above a certain quota are declared "elected".
"Surplus" (or "wasted") votes are then transferred to other candidates according to a formula.
(There are several competing formulas that are popular.)
The process is repeated until positions are deemed filled.

Variants: There are many variants, depending on how the "wasted" votes get reallocated. The *Meek method* is favored by many, but has been rejected by Scotland and BC in favor of methods that could be tabulated by hand. STV-Meek was recently adopted by New Zealand.

Factoids: Attributed to Thomas Hare in 1857, and advocated by John Stuart Mill. Described recently by Sir Michael Dummett as "The second worst system ever devised."[1]

Some 'Pros': 1. Assures both proportional representation and minority rights

2. Top choice of many experts on voting methods

3. Works well if filling more than one seat/making more than one choice

Some 'Cons': 1. Difficult to understand, and computationally complex

2. Fails Condorcet criterion.

3. Fails Monotonicity criterion.

4. Susceptible to: Participation (no-show) paradox, quasi-chaos

[1] Another quote, from Bill Tieleman: "The short version of criticism of STV is that it is complicated, confusing, prone to errors and delay, and not truly proportional, and that it reduces local accountability, increases party control, and allows special interests to dominate party nominations."

Hare STV: (Just "Hare method" in our text)

Voters rank candidates, and their highest-ranked candidates compete using a plurality vote. If there is no majority winner, the bottom choice/candidate is removed and the results tabulated again. The process is repeated until there is a majority candidate.

This can be done with multiple elections, but normally is done by ranking all choices on ballots.

Question: How many possible rankings are there? (Already asked this question in the case of the dessert example.)

Dessert example once again:

14	ABCD
10	CBDA
8	BDCA
4	DCBA
1	CDBA

First round: A:14, C:11, D:8, B:4. A has plurality of 14, no majority. B has fewest first-place votes, gets eliminated (!)

14	ACD
10+1=11	CDA
8+4=12	DCA

Second round: A:14, C:11, D:12. A has plurality of 14, no majority. C has fewest first-place votes, gets eliminated

14	AD
11+12=23	DA

Third round: A:14, D:23. D is the winner. Yum, doughnuts!

Numerical (or *Cardinal*) methods

Other voting systems have been developed which are not based on ordinal rankings but which are instead based on giving some kind of numerical weights to the candidates.

Can be seen as a variation of Borda method.

Approval voting:

Approval voting is one of the simplest and most popular such methods

Used by the Mathematical Association of America, the Institute for Operations Research and the Management Sciences, the American Statistical Association, and many corporate boards.

Idea: Votors are given a list of candidates/choices, marks some subset of them as "approved". (In other words, scores each candidate as either 0 or 1.)

The choices are then ranked by total number of "approved" votes.

Dessert example:

Instead of ranking the desserts, the 37 diners decide which ones are “acceptable” to them.

Suppose this is the tally:

Diner->	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	Total
Apple Pie	1	1	0	1	1	0	1	0	1	0	0	1	0	1	0	0	1	0	1	0	1	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	14
Biscotti	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	0	1	1	1	1	34
Cake	1	0	1	1	1	1	1	1	0	1	1	1	0	1	0	1	1	1	1	1	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	31
Doughnuts	0	0	1	0	0	0	0	1	0	1	1	0	1	0	1	1	0	1	0	1	0	1	1	0	0	1	1	0	1	1	1	1	1	1	1	0	1	21

Then Cake is the winner.