The Definite Integral

1 Area Function

Let f be a real-valued function defined on a set I; an *area function* (for f) should assign, to every $a \leq b$ in I, a value $A_f(a,b)$ satisfying the properties:

1. If $a < c < b$ then $A_f(a,b) = A_f(a,c) + A_f(c,b)$
2. If $m \leq f(x) \leq M$ for all $x \in (a,b)$ then $m(b-a) \leq A_f(a,b) \leq M(b-a)$

(Draw pictures illustrating these properties.)

For nonnegative f, $A_f(a,b)$ corresponds to our intuitive notion of the area between the graph of f and the $x-axis$ between $x = a$ and $x = b$. This extends to functions f that take negative values by simply counting area under the axis as negative area.

We will show that for a continuous function f on an interval I, there exists a unique area function for f on I. This easily extends to functions with only finitely many jump discontinuities, since if f has such a discontinuity at $x = c$ then near c, $|f|$ is bounded by some positive number M, and $|A_f(c-\epsilon, c+\epsilon)| \leq 2M\epsilon$ which goes to 0 as $\epsilon \to 0$.

2 Definition(s) of the definite integral

Let f be a function defined on an interval $[a,b]$. For the time being, let's assume that f as a continuous, positive function. We will define $A_f(a,b)$ by approximating the area by rectangles. There are several ways we can do this; it turns out that they are all the same when f is continuous, or even piecewise continuous.

2.1 Riemann sums

Subdivide the interval $[a,b]$ into n pieces, not necessarily the same size:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

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The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a *partition* of $[a,b]$.

Let $\|P\|$ be the *norm* of the partition, that is, the width of the largest subinterval:

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}\}$$

Sometimes we write $\Delta x_i = x_i - x_{i-1}$, in which case

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$$

For example, if $P = P_n$ is the *uniform partition* where the subintervals are all the same width, then $\Delta x_i = \Delta x = \frac{b-a}{n}$ for all i, and $x_i = a + \frac{b-a}{n} i$.

Pick an arbitrary point x_i^* from each interval $[x_{i-1}, x_i]$

Form the *Riemann Sum*:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

This is the area of a rectangular approximation: (picture)

There are many ways we can choose x_i^*. We could, for example, always choose $x_i^* = x_{i-1}$ (left endpoint), or $x_i^* = x_i$ (right endpoint).

If f is continuous, then it attains its minimum m_i and maximum M_i on every interval $[x_{i-1}, x_i]$.

If x_i^* is the point at which $f(x_i^*) = m_i$, then the sum

$$L_f(P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} m_i \Delta x_i$$

is called the *Riemann lower sum* corresponding to the partition P, and corresponds to approximating the area from below by rectangles.

If x_i^* is the point at which $f(x_i^*) = M_i$, then the sum

$$U_f(P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i$$

is called the *Riemann upper sum* corresponding to the partition P, and corresponds to approximating the area from above by rectangles.

Note: If P is any partition, then $L_f(P) \leq$any other Riemann sum w/r to $P \leq U_f(P)$.

Example: $f(x) = x^2$ on $[a, b], a \geq 0$. Note $L_f(P)$ is the same as a left-endpoint sum, while $U_f(P)$ is the same as a right-endpoint sum, no matter what the partition is. For simplicity, assume a uniform partition

Definition of $\int_a^b f(x)dx$: Given $a < b$ and a function $f : [a, b] \to \mathbb{R}$,

suppose $\lim\limits_{\|P\|\to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$ exists. This means that there is some unique number L such that by taking the partition P fine enough, the Riemann sum can be made arbitrarily close to L regardless of how we choose x_i^*.

Rigorously, $\lim\limits_{\|P\|\to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i = L$ provided that for every $\epsilon > 0$ there is a $\delta > 0$ such that for *any* partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and *any* choice of $x_i^* \in [x_{i-1}, x_i]$, we have

$$|\sum_{i=1}^{n} f(x_i^*)\Delta x_i - L| < \epsilon$$

For a given f and $a < b$, if the above limit exists then we call f *integrable* on $[a, b]$ and write $\int_a^b f(x)dx =$"the (definite) integral of f from a to b" for this limit.

This definition makes sense even if f is not continuous or nonnegative, though not every such function will be integrable:

Example: Suppose $f(x) = \begin{cases} 2 & \text{if } x \text{ is rational;} \\ 3 & \text{if } x \text{ is irrational} \end{cases}$. Then $L_f(P) = 14 < 21 = U_f(P)$ for any partition P of $[0, 7]$, so f is not integrable on $[0, 7]$ (or any other nontrivial interval).

Theorem 2.1 *If f is continuous on an interval $[a, b]$, or even has finitely many jump discontinuities, then f is integrable. In the case of such f all the following integrals agree:*

1. *Riemann integral*
2. *Integral defined w/r to uniform partitions (Most Calc books)*
3. *Integral defined w/r to inner and outer rectangular approximations (Darboux)*
4. *Integral defined w/r to left-endpoints, right endpoints, or midpoints as choice of x_i^* (eg, left endpoints=Cauchy)*

Note: These ways of defining the integral might *not* agree for functions which are not continuous. For example, if $[a, b] = [0, 1]$ and we always assume uniform partitions and use left endpoints, then the function in the last example would be integrable. (**Exercise:** What would be the integral in this case?)

Example $\int_a^b k\,dx$, k a constant, $a < b$

Example $\int_0^b x\,dx$, $0 < b$

Example $\int_0^b x^2 dx = \frac{b^3}{3}$, $0 < b$

Theorem 2.2 *The definite integral is an area function.*

In fact, $\int_a^b f(x)dx$ satisfies the following properties:

A. **Basic properties.** Suppose $a < b$, and $\int_a^b f(x)dx$, $\int_a^b g(x)dx$ exist on $[a, b]$

1) If $a < c < b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
2) If $m \leq f(x) \leq M$ on (a, b) then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$
3) If k is a constant then $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4) $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

B. **Extensions**

1) If $a < b$ *define* $\int_b^a f(x)dx$ to be $-\int_a^b f(x)dx$
2) *Define* $\int_a^a f(x)dx := 0$
3) **Remark:** All 4 properties above still hold if $b < a$ (and $b < c < a$ in #1)

C. **More useful properties**

1) If $f(x) \geq 0$ on (a, b) then $\int_a^b f(x)dx \geq 0$
2) If $f(x) \geq g(x)$ on (a, b) then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
3) $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$