Integral

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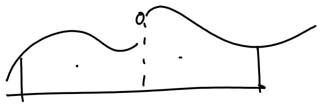
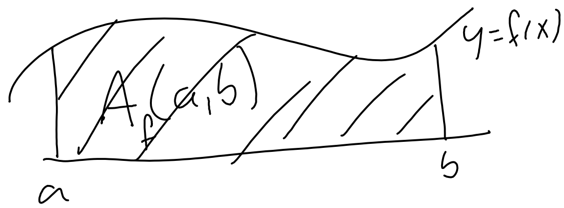
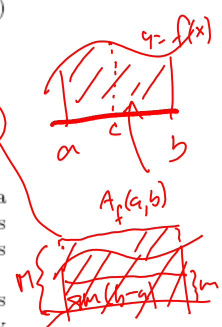




1 Area Function

Let f be a real-valued function defined on a set I; an *area function* (for f) should assign, to every $a \leq b$ in I, a value $A_f(a,b)$ satisfying the properties:

1. If $a < c < b$ then $A_f(a,b) = A_f(a,c) + A_f(c,b)$
2. If $m \leq f(x) \leq M$ for all $x \in (a,b)$ then $m(b-a) \leq A_f(a,b) \leq M(b-a)$



(Draw pictures illustrating these properties.)

For nonnegative f, $A_f(a,b)$ corresponds to our intuitive notion of the area between the graph of f and the $x-axis$ between $x = a$ and $x = b$. This extends to functions f that take negative values by simply counting area under the axis as negative area.

We will show that for a continuous function f on an interval I, there exists a unique area function for f on I. This easily extends to functions with only finitely many jump discontinuities, since if f has such a discontinuity at $x = c$ then near c, $|f|$ is bounded by some positive number M, and $|A_f(c-\epsilon, c+\epsilon)| \leq 2M\epsilon$ which goes to 0 as $\epsilon \to 0$.

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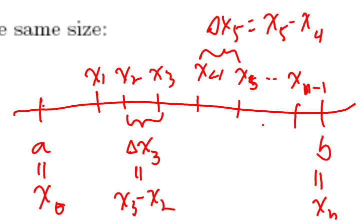
2 Definition(s) of the definite integral

Let f be a function defined on an interval $[a,b]$. For the time being, let's assume that f as a continuous, positive function. We will define $A_f(a,b)$ by approximating the area by rectangles. There are several ways we can do this; it turns out that they are all the same when f is continuous, or even piecewise continuous.

2.1 Riemann sums

Subdivide the interval $[a,b]$ into n pieces, not necessarily the same size:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$



The set $P = \{x_0, x_1, \ldots, x_n\}$ is called a *partition* of $[a,b]$.

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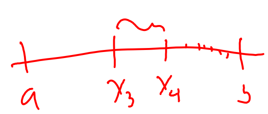


Let $\|P\|$ be the *norm* of the partition, that is, the width of the largest subinterval:

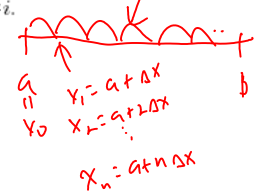
$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, x_3 - x_2, \ldots, \underbrace{x_n - x_{n-1}}_{\Delta x_n}\}$$

Sometimes we write $\Delta x_i = x_i - x_{i-1}$, in which case

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$$

For example, if $P = P_n$ is the *uniform partition* where the subintervals are all the same width, then $\Delta x_i = \Delta x = \frac{b-a}{n}$ for all i, and $x_i = a + \frac{b-a}{n} i$. $= a + i\Delta x$

n subintervals size $\frac{b-a}{n}$



Pick an arbitrary point x_i^* from each interval $[x_{i-1}, x_i]$

Form the *Riemann Sum*:

$$\sum_{i=1}^{n} \underbrace{f(x_i^*)\Delta x_i}_{\text{area of } i\text{th rectangle}}$$

This is the area of a rectangular approximation: (picture)

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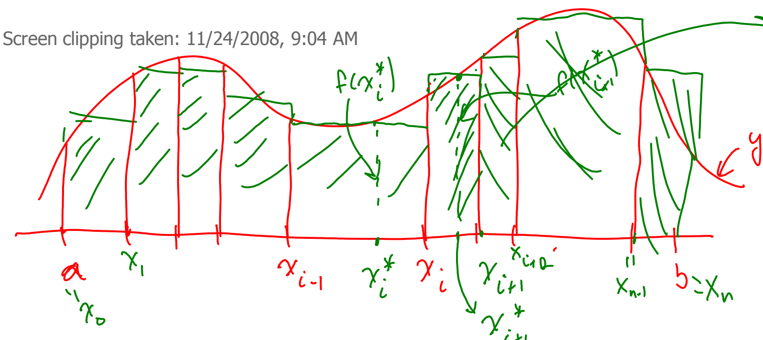


x_{i+1}^* = left endpt = pt where f is minimized on interval

x_n^* = left endpt = pt where f is maximized on $[x_{n-1}, x_n]$

There are many ways we can choose x_i^*. We could, for example, always choose $x_i^* = x_{i-1}$ (left endpoint), or $x_i^* = x_i$ (right endpoint).

If f is continuous, then it attains its minimum m_i and maximum M_i on every interval $[x_{i-1}, x_i]$.

If x_i^* is the point at which $f(x_i^*) = m_i$, then the sum

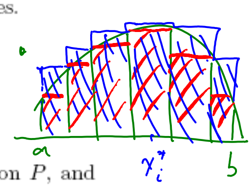
$$L_f(P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} m_i \Delta x_i$$

is called the *Riemann lower sum* corresponding to the partition P, and corresponds to approximating the area from below by rectangles.

If x_i^* is the point at which $f(x_i^*) = M_i$, then the sum

$$U_f(P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i$$

(outer)



is called the *Riemann upper sum* corresponding to the partition P, and corresponds to approximating the area from above by rectangles.

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Note: If P is any partition, then $L_f(P) \leq$any other Riemann sum w/r to $P \leq U_f(P)$.

Example: $f(x) = x^2$ on $[a, b], a \geq 0$. Note $L_f(P)$ is the same as a ~~right~~ left-endpoint sum, while $U_f(P)$ is the same as a ~~left~~ right-endpoint sum, no matter what the partition is. For simplicity, assume a uniform partition

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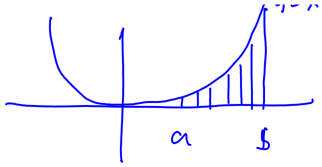
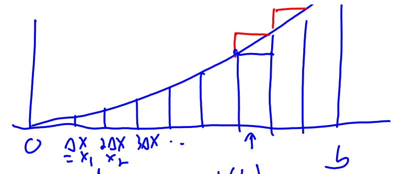
$(u = x^2)$





$\Delta x = \frac{b}{n}$, $x_i = i\left(\frac{b}{n}\right) = i\Delta x$

f increasing on $[a,b]$,
so for a lower sum, $x_i^* =$ left endpoint $= x_{i-1} = (i-1)\Delta x$

$$L_f(P_n) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} [(i-1)\Delta x]^2 \Delta x$$

$$= \sum_{i=1}^{n} (i-1)^2 (\Delta x)^3 = (\Delta x)^3 \left(\sum_{i=1}^{n} (i-1)^2\right) = (\Delta x)^3 \sum_{i=1}^{n-1} i^2$$

$$= (\Delta x)^3 \frac{(n-1)(n)(2(n-1)+1)}{6} = \frac{(b)^3}{n^3} \frac{2n^3 + \cdots}{6}$$

$$= \underbrace{\frac{b^3}{6}\left(2 + \frac{\text{quadratic}}{6n^3}\right)}_{\frac{b^3}{3} + \frac{\cdots}{n}} \longrightarrow \frac{b^3}{3} \text{ as } n \to \infty$$

$$U_f(P) = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} (i\Delta x)^2 \Delta x = (\Delta x)^3 \sum_{i=1}^{n} i^2$$

$$= \left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{b^3}{3} + \frac{\text{quadratic in } n}{n^3}$$

also $\to b^3/3$ as $n \to \infty$

$\boxed{\text{Definition of } \int_a^b f(x)dx:}$ Given $a < b$ and a function $f : [a,b] \to \mathbb{R}$,

suppose $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ exists. This means that there is some unique number L such that by taking the partition P fine enough, the Riemann sum can be made arbitrarily close to L regardless of how we choose x_i^*.

Rigorously, $\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = L$ provided that for every $\epsilon > 0$ there is a $\delta > 0$ such that for *any* partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and *any* choice of $x_i^* \in [x_{i-1}, x_i]$, we have

δ–ϵ def

$$\left|\sum_{i=1}^{n} f(x_i^*)\Delta x_i - L\right| < \epsilon$$

f Riemann Integrable

For a given f and $a < b$ if the above limit exists then we call f *integrable* on $[a,b]$ and write $\int_a^b f(x)dx$ = "the (definite) integral of f from a to b" for this limit.

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limits of integration

"integrand"

$$\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$$

same variable

(variable of integration)

This definition makes sense even if f is not continuous or nonnegative, though not every such function will be integrable:

Example: Suppose $f(x) = \begin{cases} 2 & \text{if } x \text{ is rational;} \\ 3 & \text{if } x \text{ is irrational} \end{cases}$. Then $L_f(P) = 14 < 21 = U_f(P)$ for any partition P of $[0,7]$, so f is not integrable on $[0,7]$ (or any other nontrivial interval).

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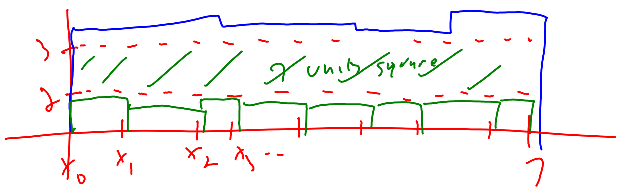


If $x_i^* \in [x_{i-1}, x_i)$ is rational, $f(x_i^*) = m_i = 2$

If $x_i^* \in [x_{i-1}, x_i)$ is irrational, $f(x_i^*) = M_i = 3$

$L_f(P) = \sum m_i \Delta x_i = \sum 2\Delta x_i = 2\sum_{i=1}^{n} \underbrace{\Delta x_i}_{\text{length of } i^{th} \text{ interval}} = 2(7-0) = 14$

$U_f(P) = \sum 3\Delta x_i = 3\sum \Delta x_i = 3(7-0) = 21 \neq 14$

True no matter how fine our partition is

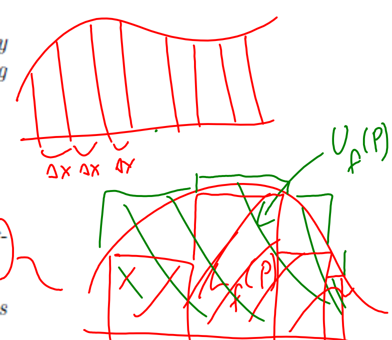
so no limit, and this f doesn't have an integral.

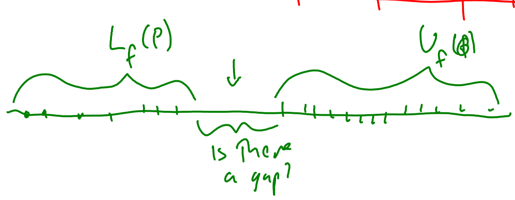
or f is increasing

or f decreasing

Theorem 2.1 *If f is continuous on an interval $[a,b]$, or even has finitely many jump discontinuities, then f is integrable. In the case of such f all the following integrals agree:*

1. *Riemann integral*
2. *Integral defined w/r to uniform partitions (Most Calc books)*
3. *Integral defined w/r to inner and outer rectangular approximations (Darboux)*
4. *Integral defined w/r to left-endpoints, right endpoints, or midpoints as choice of x_i^* (eg, left endpoints=Cauchy)*





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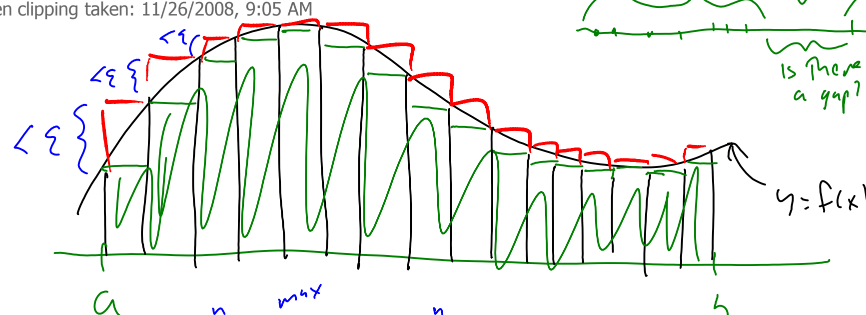

Fix $\varepsilon > 0$

f doesn't vary by more than ε on any

$U_f(P) - L_f(P) = \sum_{i=1} M_i \Delta x_i - \sum_{i=1} \underbrace{m_i}_{\min} \Delta x_i$ subinterval

$= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \sum_{i=1}^{n} \varepsilon \Delta x_i = \varepsilon \sum_{i=1} \Delta x_i = \varepsilon(b-a) \to 0$ as $\varepsilon \to 0$

Note: These ways of defining the integral might *not* agree for functions which are not continuous. For example, if $[a, b] = [0, 1]$ and we always assume uniform partitions and use left endpoints, then the function in the last example would be integrable. (**Exercise:** What would be the integral in this case?)

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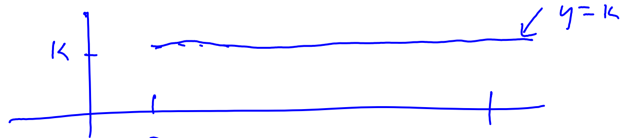
Example $\int_a^b k\,dx$, k a constant, $a < b$



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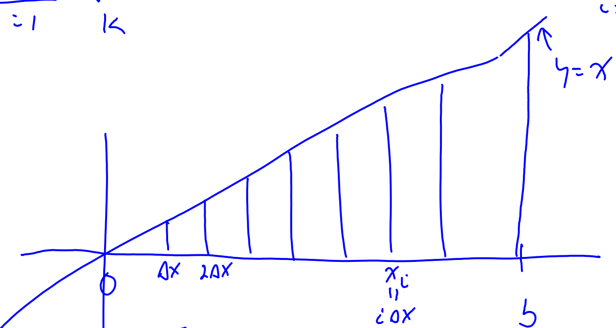
Let $P = \{x_0, \dots, x_n\}$ be any partition, x_i^* arbitrary in $[x_{i-1}, x_i]$

Riemann sum is $\sum_{i=1}^{n} \underbrace{f(x_i^*)}_{k} \Delta x_i = \sum k \Delta x_i = k \sum_{i=1}^{n} \Delta x_i = \boxed{k(b-a)}$

$\sum \Delta x_i = \cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + x_3 - \cancel{x_2} + \cdots + x_n - x_{n-1}$
$= x_n - x_0 = b - a$

Example $\int_0^b x\,dx$, $0 < b$
$= b^2/2$



Let $P = P_n$ = unif partition $= \{0, \Delta x, 2\Delta x, 3\Delta x, \dots, b\}$ where $\Delta x = \frac{b}{n}$

Let x_i^* = right endpt$_n$ $= i\Delta x$

$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1} \underbrace{f(i\Delta x)}_{i\Delta x} \Delta x = \sum_{i=1}^{n} i \underbrace{(\Delta x)^2}_{\frac{b^2}{n^2}} = \frac{b^2}{n^2} \sum_{i=1}^{n} i$

$= \frac{b^2}{n^2}\left(\frac{n^2+n}{2}\right) = \frac{b^2}{2}\left(1 + \frac{1}{n}\right) \to \frac{b^2}{2}$ as $n \to \infty$

$\cancel{\sum_{i=1}^{n} i^m = ?}$

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Example $\int_0^b x^2 dx = \frac{b^3}{3}$, $0 < b$

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Theorem 2.2 *The definite integral is an area function.*

In fact, $\int_a^b f(x)dx$ satisfies the following properties:

A. **Basic properties.** Suppose $a < b$, and $\int_a^b f(x)dx$, $\int_a^b g(x)dx$ exist on $[a, b]$

1) If $a < c < b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
2) If $m \leq f(x) \leq M$ on (a, b) then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

area fn

3) ~~If k is a constant then $\int_a^b kf(x)dx = k\int_a^b f(x)dx$~~
4) $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Prf Let P be any partition of $[a,b]$, $x_i^* \in [x_{i-1}, x_i]$ arbitrary

$$m \leq f(x_i^*) \leq M$$

$$\sum_{i=1}^n f(x_i^*)\Delta x_i \leq \sum_{i=1}^n M\Delta x_i = M(b-a)$$

$$\sum_{i=1}^n f(x_i^*)\Delta x_i \geq \sum_{i=1}^n m\Delta x_i = m(b-a)$$

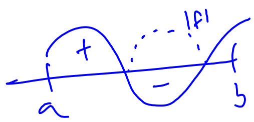
$$\sum_{i=1}^n f(x_i^*)\Delta x_i \to \int_a^b f\,dx \text{ as } \|P\| \to 0 \quad \checkmark$$

B. **Extensions**

1) If $a < b$ *define* $\int_b^a f(x)dx$ to be $-\int_a^b f(x)dx$
2) *Define* $\int_a^a f(x)dx := 0$
3) **Remark:** All 4 properties above still hold if $b < a$ (and $b < c < a$ in #1)

C. **More useful properties**

1) If $f(x) \geq 0$ on (a, b) then $\int_a^b f(x)dx \geq 0$
2) If $f(x) \geq g(x)$ on (a, b) then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
3) $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$



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C1 $f \geq 0$, $\displaystyle\sum_{i=1}^n \underbrace{f(x_i^*)}_{\geq 0}\Delta x_i \geq \sum_{i=1}^n 0\,\Delta x_i = 0$

$$\sum_{i=1}^n f(x_i^*)\Delta x_i \to \int_a^b f(x)dx \text{ as } \|P\| \to 0$$

C2 $\displaystyle\int_a^b (f-g)\,dx \geq 0$

$$\int_a^b (f-g)\,dx = \int_a^b f\,dx + \int_a^b (-1)g\,dx = \int_a^b f(x)dx - \int_a^b g(x)dx \geq 0$$

$$\therefore \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

C3 $\displaystyle\left|\int_a^b f(x)dx\right| \leftarrow \left|\sum_{i=1}^n f(x_i^*)\Delta x_i\right| \leq \sum_{i=1}^n |f(x_i^*)\Delta x_i|$ since $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$

$$= \sum_{i=1}^n |f(x_i^*)|\Delta x_i \to \int_a^b |f|\,dx$$