# Ordinals - summary

Here is everything you really need to know about ordinals:

1. The definition: An *ordinal number* is a set $a$ such that (i) the elements of $a$ are well-ordered (by $\epsilon$), and (ii) (transitivity) If $x \in y \in a$ then $x \in a$
2. It follows that every *element* of an ordinal $a$ is also a *subset* of $a$, and is in fact an ordinal itself.
3. Ordinals satisfy trichotomy, so any set of ordinals is linearly ordered.
4. There is a first ordinal (namely, $\emptyset$), but no last ordinal: in fact, if $a$ is an ordinal, so is $a \cup \{a\}$ (show this!), which we call the *successor* of $a$, and denote by $a^+$. Note there is no ordinal strictly between $a$ and $a^+$, so $a^+$ is the *immediate* successor of $a$.
5. Any set of ordinals is well ordered; any nonempty collection of ordinals has a least element.
6. The natural numbers are (finite) ordinals:

   $$0 = \emptyset, 1 = \{\emptyset\} = \{0\}, 2 = \{0, 1\}, \ldots, n + 1 = \{0, 1, \ldots, n\}, \ldots$$

7. The *set* of natural numbers is also an ordinal, whih we denote by $\omega$ (so "$n < \omega$" is just a strange way of writing "$n \in \mathbb{N}$"). $\omega$ is the least infinite ordinal, and is of course a *countable ordinal*.
8. There are many other countable ordinals. Examples include $\omega^+$, $\omega + \omega$ (whatever that is), etc.
9. The first uncountable ordinal is denoted by $\omega_1$. To show that such an ordinal even exists requires a bit of work! In fact, $\omega_1$ is just the set of all countable ordinals.

10. Every well-ordered set is order-isomorphic to a unique ordinal.

11. By the Axiom of Choice, every set $X$ can be well-ordered, so has the same cardinality as some ordinal. The least such ordinal is the *cardinality* of $X$, or card($X$). The fact that every set has a cardinality is actually equivalent to AC. If $\kappa = \text{card}(X)$ then we can write $X = \{x_i\}_{i<\kappa}$, where all the elements $x_i$ are distinct.

12. (Extremely Useful!) If $\{a_n\}_{n<\omega}$ is a set of countable ordinals, then for some $\beta < \omega_1$, $\forall n < \omega \;\; a_n < \beta$.