Math 671 - Assignment 6 - Due October 16

1. (Don't hand in.) If $p > 0$ and $X \in \mathcal{L}^p$ then $x^p P(X > x) \to 0$ as $x \to \infty$

2. If X, Y are independent and $X + Y \in \mathcal{L}^p$ $(p > 0)$ then $X \in \mathcal{L}^p$. (Hint: Show that for large enough $\lambda > 0$,

$$P(|X| > \lambda) \leq 2P(|X| > \lambda, |Y| < \lambda/2) \leq 2P(|X + Y| > \lambda/2)$$

and then use a result I said in class we wouldn't use very much (if at all).

3. Suppose X is a nonnegative random variable with $X \in \mathcal{L}^p$ for all $p > 0$. Define $g(p) = \ln \mathbb{E}(X^p), 0 < p < \infty$. Prove that g is a convex function on $(0, \infty)$. (Hint: Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$, let $p = 1/\alpha, q = 1/\beta$, note $p, q > 1$, and use Hölder's inequality.)

4. This problem sketches an alternate proof of Hölder's inequality. Do not hand in part (a).

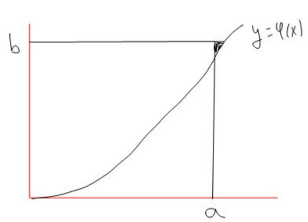
   (a) Let $\phi : [0, \infty) \to [0, \infty)$ be continuous and strictly increasing, with $\phi(0) = 0$. Let $\psi = \phi^{-1}$. Prove that for $a, b > 0$, $ab \leq \int_0^a \phi(x)dx + \int_0^b \psi(y)dy$. (See picture below, sorry about its quality.)

   (b) Conclude that if $p, q > 1$ are conjugate exponents and $a, b > 0$ then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

   For the final two parts, assume that $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$, where $p, q > 1$ are conjugate exponents.

   (c) Prove that if $\|f\|_p = \|g\|_q = 1$ then $\mathbb{E}(|fg|) \leq \|f\|_p \|g\|_q$. (Hint: use the previous part.)

   (d) Finally, prove Hölder's theorem for general $f \in \mathcal{L}^p$, $g \in \mathcal{L}^q$. (Hint: divide f and g by something suitable.)



Note $\int_0^a \varphi(x)dx + \int_0^b \psi(y)dy$ includes both the rectangle (area = ab) plus the shaded part.