MATH 611 **Due Friday December 4, 1992**

1. Prove that a ring R is local if and only if for every $r \in R$, either r or $1-r$ is invertible.

2. An integer is *square-free* if it is not divisible by any integer of the form k^2 for $k > 1$. For instance, 30 and 35 are square free, but 12 and 45 are not.
Let M be the set of all rational numbers with square-free denominators. Prove that M is a $\mathbb{Z}$-submodule of $\mathbb{Q}$ and that M is generated by the set of fractions $\frac{1}{p}$ such that p is a prime number.

3. Let $R = \{(x, y) \in \mathbb{Z} \oplus \mathbb{Z} \mid y \equiv x \pmod{13}\}$.
For any integer k not divisible by 13, let $M_k = \{(x, y) \in \mathbb{Z} \oplus Z \mid y \equiv kx \pmod{13}\}$.

a) Prove that M_k is an R-module and that if $M_k = K \oplus N$ then

$$\{\,(13x, 0) \mid x \in \mathbb{Z}\} \subseteq K \quad \text{or} \quad \{\,(13x, 0) \mid x \in \mathbb{Z}\} \subseteq N.$$

Likewise

$$\{\,(0, 13y) \mid y \in \mathbb{Z}\} \subseteq K \quad \text{or} \quad \{\,(0, 13y) \mid y \in \mathbb{Z}\} \subseteq N.$$

Conclude that M_k is an indecomposable R-module.
(Recall that if $M_k = K \oplus N$ then $N \approx M_k/K$.)

b) Prove that if $\varphi\colon M_k \to M_\ell$ is an isomorphism, then

(1) $\varphi(13x, 0) = (\pm 13x, 0)$
(2) $\varphi(0, 13y) = (0, \pm 13y)$
(3) $\varphi(x, y) = (\pm x, \pm y)$.

Conclude that $M_k \approx M_\ell$ if and only if $\ell \equiv \pm k \pmod{13}$.
(Note that $M_1 = R$.)

c) Let $P = \{\,(x, y, z) \in \mathbb{Z}^3 \mid y \equiv x \pmod{13}\}$.
Make this into an R-module by defining $(a, b)(x, y, z) = (ax, by, az)$ for $(a, b) \in R$.
Prove that $P \approx R \oplus L$ where L is the left ideal $\{(13x, 0) \mid x \in \mathbb{Z}\} \subseteq R$.

d) Now for any k with $k \not\equiv 0 \pmod{13}$, let

$$M'_k = \{\,(kz, y, z) \in \mathbb{Z}^3 \mid y \equiv kz \pmod{13}\} \subseteq P.$$

Prove that $M'_k \approx M_k$.

e) Now since $k \not\equiv 0 \pmod{13}$, there exists ℓ with $k\ell \equiv 1 \pmod{13}$, say $k\ell - 13n = 1$ for some $n \in \mathbb{Z}$. Let

$$L' = \{\,(13\,nz, 0, \ell z) \mid x \in \mathbb{Z}\}.$$

Prove that L' is an R-submodule of $\mathbb{Z}^3$ and $L' \approx L$.

f) Prove that $R \oplus L \approx P = M'_k \oplus L' \approx M_k \oplus L$.
(As previously noted, if $k \not\equiv \pm 1 \pmod{13}$ then $M_k \not\approx R$.)

g) Prove that for any $k, \ell \not\equiv 0 \pmod{13}$, $M_k \oplus M_\ell \approx R \oplus M_{k\ell}$.

3. f) Recall that

$$\begin{aligned} P &= \{\,(x, y,\, z) \in \mathbb{Z}^3 \mid y \equiv x \,(\mathrm{mod}\ 13)\}, \\ M_k' &= \{\,(kz, y,\, z) \in \mathbb{Z}^3 \mid y \equiv kz \,(\mathrm{mod}\ 13)\} \ \subseteq P, \\ L' &= \{\,(13\,nz, 0,\, \ell z) \mid x \in \mathbb{Z}\} \ \subseteq P, \\ \text{and} \quad & k\ell - 13n = 1. \end{aligned}$$

Prove that $R \oplus L \approx P = M_k' \oplus L' \approx M_k \oplus L$.
(This is how the problem should have read. $M_k' \oplus L'$ is an *internal* direct sum, the others are external direct sums.)

Note first that $M_k' + L' \subseteq P$. Furthermore $M_k' \cap L' = 0$ since if $(kz, y, z) \in M_k' \cap L'$ then $y = 0$ and $(1 + 13n)z = k\ell z = 13knz$, which forces $z = 0$ since otherwise $1 = 13n(k-1)$, which is impossible.
It remains to be seen that $M_k' + L' = P$. It suffices to see that $(x, y, 0) \in M_k' + L'$ whenever $y \equiv x \,(\mathrm{mod}\ 13)$ and that $(0, 0, 1) \in M_k' + L'$.
But

$$(x, y, 0) = (k\ell x,\, k\ell y - 13ny,\, \ell x) - (13nx, 0, \ell x) \in M_k' + L'$$

and

$$(0, 0, 1) = -(13kn, 0, 13n) + k\,(13n, 0, \ell) \in M_k' + L'.$$

g) Prove that for any $k,\, \ell \not\equiv 0 \pmod{13}$, $M_k \oplus M_\ell \approx R \oplus M_{k\ell}$.

Note first that the result is clear if $k = 1$ or $\ell = 1$. Furthermore, if $k' \equiv \pm k \pmod{13}$ then $M_{k'} \approx M_k$. It follows that there is no loss of generality in assuming that k and ℓ are relatively prime. (In fact, by Dirichlet's Theorem on primes in arithmetic progressions, one can choose distinct primes for k and ℓ. Alternatively, one can choose $2 \le k,\, \ell \le 6$ and simply by considering cases see that if k and ℓ have a common factor then k and $13 - \ell$ do not.) Thus there exist integers s and t with

$$sk + t\ell = 1.$$

Now let

$$\tilde{M}_k = \{\, (kx, y, tx, t\ell y) \in \mathbb{Z}^4 \mid y \equiv kx \pmod{13}\} \subseteq R \oplus M_{k\ell}.$$

One easily checks that there is an isomorphism θ from M_k to $\tilde{M}_k$ given by $\theta(x, y) = (kx, y, tx, t\ell y)$. Likewise let

$$\tilde{M}_\ell = \{\, (\ell x, y, -sx, -sky) \in \mathbb{Z}^4 \mid y \equiv \ell x \pmod{13}\} \subseteq R \oplus M_{k\ell}.$$

Then $\tilde{M}_\ell \approx M_\ell$. It is routine to see that $\tilde{M}_k \cap \tilde{M}_\ell = 0$.
To see that $\tilde{M}_k + \tilde{M}_\ell = R \oplus M_{k\ell}$, it suffices to see that whenever $y \equiv x \pmod{13}$ then $(x, y, 0, 0) \in \tilde{M}_k + \tilde{M}_\ell$ and whenever $y \equiv k\ell x \pmod{13}$ then $(0, 0, x, y) \in \tilde{M}_k + \tilde{M}_\ell$. In fact, if $y \equiv x \pmod{13}$ then

$$(x, y, 0, 0) = s\,(kx, ky, tx, t\ell ky) + t\,(\ell x, \ell y, -sx, -sk\ell y) \in \tilde{M}_k + \tilde{M}_\ell.$$

and if $y \equiv k\ell x \pmod{13}$ then

$$(0, 0, x, y) = (k\ell x, y, t\ell x, t\ell y) + (-k\ell x, -y, skx, sky) \in \tilde{M}_k + \tilde{M}_\ell.$$

Note: The more interesting challenge is to prove that $M_k \oplus M_\ell \approx M_{k'} \oplus M_{\ell'}$ if and only if $k\ell \equiv \pm k'\ell' \pmod{13}$. From the above, one can see that it suffices to prove that $R \oplus M_k \approx R \oplus M_{k'}$ if and only if $k \equiv \pm k' \pmod{13}$. Part **f)** should suggest that this is not at all obvious.