Lifting Idempotents Modulo A Nil Ideal

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Theorem. Let R be a ring and N an ideal such that all the elements in N are nilpotent. (Such an ideal is usually called a **nil ideal**. Do not use the term *nilpotent ideal*, because this is (usually) something different.)
If $\bar{e}$ is an idempotent element in R/N, then there exists an idempotent pre-image for $\bar{e}$.

PROOF: Choose any pre-image e for $\bar{e}$. Let $f = 1 - e$. Then **(a)** $ef = fe$ and **(b)** $e + f \equiv 1 \pmod{N}$. Now $ef = e - e^2 \in N$ since by hypothesis $e \equiv e^2 \pmod{N}$. (The images of e and e^2 in R/N are $\bar{e}$ and $\bar{e}^2$, but by hypothesis, $\bar{e}^{\,2} = \bar{e}$.) Since N consists of nilpotent elements, there exists some $k \geq 1$ such that $0 = (ef)^k = e^k f^k$.

Now e^k is also a pre-image for $\bar{e}$, since $e^k \equiv e \pmod{N}$. Replacing e and f by e^k and f^k, conditions **(a)** and **(b)** are preserved and furthermore, **(c)** $ef = fe = 0$. Now let $x = 1 - e - f \in N$ (using condition **(b)**). Then there must exist some ℓ such that $x^\ell = 0$, and it follows that $1 - x$ has an inverse $u = 1 + x + \cdots + x^{\ell-1}$. Since x commutes with e and f, we see that u commutes with e and f.

Since $x \in N$, $u \equiv 1 \pmod{N}$, and so we can replace e and f with ue and uf, and e will still be a pre-image for $\bar{e}$ and conditions **(a)**, **(b)**, and **(c)** will still hold, and furthermore it is now true that **(d)** $e+f = 1$. But then $e = e(e+f) = e^2+ef = e^2$ (since $ef = 0$ by condition **(c)**), so that this latest version of e is an idempotent pre-image for $\bar{e}$, as required. $\boxed{\checkmark}$