MATH 612 **Due Wednesday April 7, 1993**

1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in a commutative ring R. Prove that $R/\mathfrak{a} \otimes_R R/\mathfrak{b} \approx R/(\mathfrak{a}+\mathfrak{b})$.

2. **a)** Prove that if T is a torsion abelian group and D is divisible, then $T \otimes_{\mathbb{Z}} D = 0$.
b) Prove that the tensor product functor is in general not left exact by giving an example of abelian groups X and $N \subseteq M$ such that the induced map $X \otimes_{\mathbb{Z}} N \to X \otimes_{\mathbb{Z}} M$ is not a monomorphism. Thus in this case one cannot correctly consider $X \otimes_{\mathbb{Z}} N$ as a submodule of $X \otimes_{\mathbb{Z}} M$.

3. **a)** Prove that the tensor product respects arbitrary direct sums, i.e. if $\{M_i\}_{i\in I}$ is a family of left R-modules, and X is a right R-module, then $X \otimes_R \bigoplus_I M_i \approx \bigoplus_I X \otimes_R M_i$.
b) On the other hand, then tensor product usually does not respect infinite direct products. Although there is an obvious map $\zeta\colon X \otimes_R \prod_I M_i \to \prod_I X \otimes_R M_i$, if I is infinite this need not be either a monomorphism or an epimorphism. Describe this map and prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_1^\infty \mathbb{Z} \to \prod_1^\infty \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}$ is not an epimorphism. Prove that if P is the set of prime numbers then $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_P \mathbb{Z}/p\mathbb{Z} \to \prod_P \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is not a monomorphism by showing that $\prod_P \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$ but $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_P \mathbb{Z}/p\mathbb{Z}$ is non-trivial.

4. Prove that if M is a free R-module, and R' is a ring containing R, then $R' \otimes_R M$ is a free R'-module.

5. Let D be a skew-field and let R be the ring of $n \times n$ matrices over D. Let L be the set of $1 \times n$ matrices and M the set of $n \times 1$ matrices over D. Then with the usual definition of matrix multiplication, L is a right R-module and M is a left R-module . Furthermore, L is a left D-module and so $L \otimes_R M$ can be given the structure of a left D-module by defining $d(\ell \otimes m) = d\ell \otimes m$. (One checks that this is well defined.) Prove that as D-modules, $L \otimes_R M \approx D$.