

A Continued Fraction Example

Let $\alpha = \sqrt{6} - 2$. Then

$$\begin{aligned}\frac{1}{\alpha} &= \frac{1}{\sqrt{6} - 2} = \frac{\sqrt{6} + 2}{2} = \frac{\alpha + 4}{2} = 2 + \frac{\alpha}{2}, \\ \frac{2}{\alpha} &= 4 + \alpha.\end{aligned}$$

Note that $0 < \frac{\alpha}{2} < 1$. Thus $\left[\frac{1}{\alpha}\right] = 2$ and $\left[\frac{2}{\alpha}\right] = 4$.

Then we get

$$a_0 = [\alpha] = 0 \quad \text{and} \quad \alpha_1 = \frac{1}{\text{Fr}(\alpha)} = \frac{1}{\alpha} = 2 + \frac{\alpha}{2}.$$

So

$$a_1 = [\alpha_1] = 2 \quad \text{and} \quad \alpha_2 = \frac{1}{\text{Fr}(\alpha_1)} = \frac{2}{\alpha} = 4 + \alpha.$$

Then

$$a_2 = [\alpha_2] = 4 \quad \text{and} \quad \alpha_3 = \frac{1}{\text{Fr}(\alpha_2)} = \frac{1}{\alpha} = 2 + \frac{\alpha}{2}.$$

Continuing in this way, one sees that for $k \geq 0$,

$$\alpha_{2k+1} = 2 + \frac{\alpha}{2} \quad \text{and} \quad a_{2k+1} = 2$$

and for $k \geq 1$,

$$\alpha_{2k} = 4 + \alpha \quad \text{and} \quad a_{2k} = 4.$$

Thus $\alpha = [0; 2, 4, 2, 4, 2, 4, \dots]$.

As another way of seeing this, note that

$$\begin{aligned}\alpha &= \frac{1}{1/\alpha} = \frac{1}{2 + \frac{\alpha}{2}} \\ &= \frac{1}{2 + \frac{1}{2/\alpha}} = \frac{1}{2 + \frac{1}{4 + \alpha}}.\end{aligned}$$

Now take this expression for α and substitute it into itself (i. e. into the right hand side), getting

$$\begin{aligned} \alpha &= \frac{1}{2 + \frac{1}{4 + \alpha}} = \frac{1}{2 + \frac{1}{4 + \frac{1}{2 + \frac{1}{4 + \alpha}}}}} \\ &= \frac{1}{2 + \frac{1}{4 + \frac{1}{2 + \frac{1}{4 + \frac{1}{2 + \frac{1}{4 + \alpha}}}}}}}, \end{aligned}$$

etc.