The Principal Curvatures On a Surface

E. L. Lady

NOTATION: $\alpha(t)$ is a curve on a surface S. $\quad p = \alpha(0)$.

$'$ always denotes differentiation **with respect to t**.

s denotes arc length measured from p:

Thus $s' = |\alpha'| =$ "speed" (p. 6) and $\left|\dfrac{d\alpha}{ds}\right| = 1$ (Do Carmo, p. 16).

$k(t) = \left|\dfrac{d^2\alpha}{ds^2}(t)\right| =$ "curvature" (DEFINITION, DoCarmo, p. 16).

$\mathbf{n}(t) = \dfrac{1}{k(t)}\dfrac{d^2\alpha}{ds^2} =$ "unit normal vector to the curve" (DoCarmo, p. 17).

$\mathbf{N}(t) =$ the unit normal vector to S at the point $\alpha(t)$.

$k_n(t) = k(t)\,\langle \mathbf{N}(t), \mathbf{n}(t)\rangle =$ "the normal curvature of α" (DEFINITION 3, DoCarmo, p. 141).

LEMMA A. $\quad \alpha''(t) = s''(t)\dfrac{d\alpha}{ds}(t) + (s'(t))^2\, k(t)\,\mathbf{n}(t)$.

Proof. By the chain rule, $\alpha'(t) = \dfrac{d\alpha}{ds}\dfrac{ds}{dt} = s'\dfrac{d\alpha}{ds}$. Then by the product rule and chain rule, $\alpha''(t) = s''\dfrac{d\alpha}{ds} + s'\dfrac{d}{dt}\left(\dfrac{d\alpha}{ds}\right) = s''\dfrac{d\alpha}{ds} + s'\dfrac{d^2\alpha}{ds^2}\dfrac{ds}{dt} = s''\dfrac{d\alpha}{ds} + (s')^2 k\mathbf{n}$.

PROPOSITION B. $-\langle \mathbf{N}', \alpha'\rangle = \langle \mathbf{N}, \alpha''\rangle = (s')^2 k_n$. (Compare DoCarmo, p. 142.)

Proof. Since α' is a tangent vector to S and $\mathbf{N}$ is perpendicular to S, $\langle \mathbf{N}, \alpha'\rangle = 0$. Thus $0 = \dfrac{d}{dt}\langle \mathbf{N}, \alpha'\rangle = \langle \mathbf{N}', \alpha'\rangle + \langle \mathbf{N}, \alpha''\rangle$ (by the product rule), so that

$$-\langle \mathbf{N}', \alpha'\rangle = \langle \mathbf{N}, \alpha''\rangle = s''\left\langle \mathbf{N}, \frac{d\alpha}{ds}\right\rangle + (s')^2 k\langle \mathbf{N}, \mathbf{n}\rangle = 0 + (s')^2 k_n \quad \text{(using Lemma A).}$$

THEOREM C. [See p. 154.] Let $\mathbf{X}$ be a parametrization of S and let $\mathbf{N}(u,v)$ denote the normal vector to S at the point $\mathbf{X}(u,v)$. Let $e = -\langle \mathbf{N}_u, \mathbf{X}_u \rangle$, $f = -\langle \mathbf{N}_u, \mathbf{X}_v \rangle$, and $g = -\langle \mathbf{N}_v, \mathbf{X}_v \rangle$. Let $\alpha(t) = \mathbf{X}(u(t), v(t))$ be a curve on S and let $\mathbf{w} = \alpha'(0)$. Let $q = (u(0), v(0))$ and $p = \mathbf{X}(q)$. Then $(s'(0))^2 k_n(0) =$

$$= -\langle \mathbf{N}'(0), \alpha'(0) \rangle = \langle d\mathbf{N}_p(\mathbf{w}), \mathbf{w} \rangle \;=\; e(q)\, u'(0)^2 + 2f(q)\, u'(0)v'(0) + f(q)\, v'(0)^2.$$

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DEFINITION. Let S be a surface, $p \in S$, $\mathbf{N}$ the unit normal, $\mathbf{w} \in T_p(S)$. Then $d\mathbf{N}_p(\mathbf{w})$ is defined as follows: Let $\alpha(t)$ be any curve on S with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{w}$; then $d\mathbf{N}_p(\mathbf{w}) = \dfrac{d}{dt}\mathbf{N}(\alpha(t))\Big|_{t=0}$. Alternatively, we notice that $d\mathbf{N}_p(\mathbf{X}_u) = \mathbf{N}_u$ and $d\mathbf{N}_p(\mathbf{X}_v) = \mathbf{N}_v$. Thus if $\mathbf{w} = a\mathbf{X}_u + b\mathbf{X}_v$, then $d\mathbf{N}_p(\mathbf{w}) = a\mathbf{N}_u + b\mathbf{N}_v$.
We define the curvature of S at p in the direction $\mathbf{w}$ to be
$k(\mathbf{w}) = -\langle d\mathbf{N}_p(\mathbf{w}), \mathbf{w} \rangle \,/\, |\mathbf{w}|^2 = -\langle \mathbf{N}'(0), \alpha'(0) \rangle \,/\, |\mathbf{w}|^2$. This is the same as $k_n(\alpha)$, the normal curvature α at p.

LEMMA D. For any $\mathbf{w}_1, \mathbf{w}_2 \in T_p(S)$, $\langle d\mathbf{N}_p(\mathbf{w}_1), \mathbf{w}_2 \rangle = \langle d\mathbf{N}_p(\mathbf{w}_2), \mathbf{w}_1 \rangle$.

THEOREM E (p. 144). Let $\mathbf{v}, \mathbf{w} \in T_p(S)$ be vectors such that $|\mathbf{v}| = |\mathbf{w}| = 1$, $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ and $k(\mathbf{v})$ is the maximum curvature in any direction at p. THEN

(1) $d\mathbf{N}_p(\mathbf{v}) = -k_1\mathbf{v}$ and $d\mathbf{N}_p(\mathbf{w}) = -k_2\mathbf{w}$, where $k_1 = k(\mathbf{v})$ and $k_2 = k(\mathbf{w})$.

(2) $-k_1$ and $-k_2$ are the eigenvalues of the linear transformation $d\mathbf{N}_p : T_p(S) \longrightarrow T_p(S)$.

(3) k_2 is the minimum curvature in any direction at p.

(4) If $r\mathbf{v} + s\mathbf{w}$ is any vector in $T_p(S)$, then $k(r\mathbf{v} + s\mathbf{w}) = \dfrac{k_1 r^2 + k_2 s^2}{r^2 + s^2}$.

Proof of Theorem E. Since $\mathbf{v}$ and $\mathbf{w}$ form an orthogonal basis of $T_p(S)$, $d\mathbf{N}_p(\mathbf{v}) = a\mathbf{v} + b\mathbf{w}$ and $d\mathbf{N}_p(\mathbf{w}) = b'\mathbf{v} + c\mathbf{w}$. Since $\langle \mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{w}, \mathbf{w}\rangle = 1$ and $\langle \mathbf{v}, \mathbf{w}\rangle = 0$,

$$\begin{aligned} a =& \langle d\mathbf{N}_p(\mathbf{v}), \mathbf{v}\rangle = -|\mathbf{v}|^2\, k(\mathbf{v}) = -k_1, \\ c =& \langle d\mathbf{N}_p(\mathbf{w}), \mathbf{w}\rangle = -k(\mathbf{w}) = -k_2, \\ b =& \langle d\mathbf{N}_p(\mathbf{v}), \mathbf{w}\rangle = \langle d\mathbf{N}_p(\mathbf{w}), \mathbf{v}\rangle = b'. \end{aligned}$$

Now if $r\mathbf{v} + s\mathbf{w} \in T_p(S)$ then there exists a unique θ with $0 \le \theta < 2\pi$ and $r\mathbf{v}+s\mathbf{w} = \sqrt{r^2+s^2}\,(\mathbf{v}\cos\theta + \mathbf{w}\sin\theta)$. (Think in terms of polar coordinates.) Furthermore $k(r\mathbf{v} + s\mathbf{w}) = k(\mathbf{v}\cos\theta + \mathbf{w}\sin\theta) = -\langle d\mathbf{N}_p(\mathbf{v}\cos\theta + \mathbf{w}\sin\theta), \mathbf{v}\cos\theta + \mathbf{w}\sin\theta\rangle =$ $k_1\cos^2\theta + 2b\sin\theta\cos\theta + k_2\sin^2\theta$. Denote this quantity by $k(\theta)$. In other words, $k(\theta)$ is the curvature of S "in the direction θ" from $\mathbf{v}$. By choice of $\mathbf{v}$, this function has a maximum at $\theta = 0$. Thus $k'(0) = 0$. But

$$k'(0) = 2(k_2 - k_1)\sin\theta\cos\theta + 2b(\cos^2\theta - \sin^2\theta) = (k_2 - k_1)\sin 2\theta + 2b\cos 2\theta,$$

so $0 = k'(0) = 2b$. Thus $b = b' = 0$, so $d\mathbf{N}_p(\mathbf{v}) = -k_1\mathbf{v}$ and $d\mathbf{N}_p(\mathbf{w}) = -k_2\mathbf{w}$. Thus $-k_1$ and $-k_2$ are the eigenvalutes of $d\mathbf{N}_p$. (There can only be two eigenvalues since $T_p(S)$ is two-dimensional.) Furthermore, since $k_1 \ge k_2$ then for any θ, $k_2 = k_2\cos^2\theta + k_2\sin^2\theta \le k_1\cos^2\theta + k_2\sin^2\theta = k(\theta)$, so k_2 is the minimum possible curvature in any direction.

Finally, for any $r\mathbf{v} + s\mathbf{w}$ we get

$$k(r\mathbf{v} + s\mathbf{w}) = \frac{-\langle d\mathbf{N}_p(r\mathbf{v} + s\mathbf{w}), r\mathbf{v} + s\mathbf{w}\rangle}{|r\mathbf{v} + s\mathbf{w}|^2} = \frac{\langle k_1 r\mathbf{v} + k_2 s\mathbf{w},\, r\mathbf{v} + s\mathbf{w}\rangle}{r^2 + s^2} = \frac{k_1 r^2 + k_2 s^2}{r^2 + s^2}.$$