MATH 420 **FINAL EXAM** Monday December 13, 1993

PLEASE RETURN TEST SHEET

Do any seven problems.

(15) **1.** Prove that a positive integer n is prime if and only if $\varphi(n) = n - 1$.

(15) **2.** Prove that there do not exist integers x and y such that $x^2 + y^2 \equiv 3 \pmod{4}$.

(20) **3.** Prove that a positive integer m is even if and only if $\varphi(2m) = 2\varphi(m)$.

(20) **4.** Find the continued fraction expansions for the following real numbers:
a) $\sqrt{20}$
b) $42/29$

(20) **5.** Prove that if n is any natural number and s is the sum of the digits of n, then n and s give the same remainder when divided by 9.
(In other words, if $a_k, \ldots, a_0$ are the digits when n is written in its normal decimal representation and if $n = 9q + r$ with $0 \le r < 9$, then $a_k + \cdots + a_0 = 9q_2 + r$ for some q_2.)

(30) **6.** Let a be an integer and p a prime. Prove that if $x^2 - x + a$ is not a multiple of p for all integers x with $0 \le x < p$ then $x^2 - x + a$ is not a multiple of p for any x whatsoever.

(30) **7.** We know that if $d = (a, b)$ then there exist integers x and y such that $ax + by = d$. Use this to prove that the linear Diophantine equation $ax + by = c$ has a solution if and only if $(a, b) | c$.

(30) **8.** Prove that the simple continued fraction expansion for $\sqrt{a^2 - 1}$ (where $a \geq 2$) is $[\, a-1;\, \overline{1,\, 2a-2}\,]$.

(35) **9.** Let p be a prime number of the form $4k + 1$ and let q be any prime number except 2. Prove that $p | x^2 - q$ for some x if and only if $q | y^2 - p$ for some y.

(40) **10.** **a)** Prove that for any x, every prime factor of $x^2 + 1$ (except 2) has the form $4k + 1$. In particular, if $p_1, \ldots, p_n$ are prime numbers, then the odd prime factors of $p_1^2 \cdots p_n^2 + 1$ must have the form $4k + 1$.

b) Use this to prove that there are infinitely many primes of the form $4k + 1$.