A COURSE IN HOMOLOGICAL ALGEBRA

CHAPTER 11: Auslander's Proof of Roiter's Theorem

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A category $\mathcal{C}$ is **skeletally small** if there exists a **set** of objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is isomorphic to one of the objects in this set. (The category of finitely generated modules over a ring Λ is skeletally small, for instance.)

The main theorem to be proved in this chapter is as follows:

Theorem. Let $\mathcal{C}$ be a skeletally small abelian category satisfying the following conditions:

(1) Every object in $\mathcal{C}$ has finite length.

(2) $\mathcal{C}$ has only finitely many simple objects (up to isomorphism).

(3) The endomorphism ring of an indecomposable object in $\mathcal{C}$ is a local ring.

Then there exists only finitely many indecomposable objects in $\mathcal{C}$ (up to isomorphism) if and only if there is a bound on the lengths of indecomposable objects in $\mathcal{C}$.

Note that the category of finitely generated modules over an artinian ring Λ satisfies conditions (1) through (3) above.

Notation and Terminology. From here on, by "abuse of language," we will omit the words "up to isomorphism" as used above. We will use the notation $\mathcal{C}(X,Y)$ to indicate $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (also commonly denoted as $\mathrm{Mor}_{\mathcal{C}}(X,Y)$). Recall that an additive contravariant functor from $\mathcal{C}$ to $\mathcal{A}b$ is called **representable** if it is isomorphic to $\mathcal{C}(\_\_, C)$ for some object $C \in \mathcal{C}$. All functors under consideration here will be additive, although sometimes through absent-mindness or laziness this will not be explicitly stated.

A Note on Subfunctors. A **subfunctor** of a functor F is a functor E together with a **monic** natural transformation $\tau \colon E \to F$. In the most common case, when the target category for F and G is a concrete category (i. e. $F(X)$ consists of a set with some sort of additional structure), it is common to assume, without real loss of generality, that in fact for every object X in the source category, $E(X) \subseteq F(X)$ and τ_X is simply the inclusion map. The fact that the inclusion map is natural simply means that whenever $f \colon X \to Y$ then $F(f)$ restricts to a map $E(X) \to E(Y)$. In this case, one commonly writes $E \subseteq F$.

A functor F is **simple** if it has no subfunctors other than itself and the zero functor. We say that F has **finite length** n if there is a chain

$$0 \subsetneq F_1 \subsetneq F_2 \subseteq \cdots \subsetneq F_n = F$$

such that F_1 and all the quotients F_i/F_{i-1} are simple functors. The category of functors from a category $\mathcal{C}$ into the category of abelian groups $\mathcal{A}b$ or any abelian category is abelian, and hence the Jordan-Hölder Theorem applies, showing that the length of a functor is well defined.

Lemma 1 [Yoneda Lemma]. If F is an additive contravariant functor from $\mathcal{C}$ into the category of abelian groups $\mathcal{A}b$, then for any object $C \in \mathcal{C}$, $\mathrm{Nat}(\,\mathcal{C}(\_\_, C),\, F\,)$ is naturally isomorphic to $F(C)$, where the isomorphism is given by $\tau \mapsto \tau_C(1_C)$.

Auslander's idea is to study $\mathcal{C}$ by using the Yoneda embedding from $\mathcal{C}$ into the category of additive contravariant functors from $\mathcal{C}$ into $\mathcal{A}b$, which maps an object C into the contravariant functor $\mathcal{C}(\_\_, C)$. This is a full embedding, but is not usually exact (essentially because Hom is not a right exact functor). In fact, the functor $\mathcal{C}(\_\_, C)$ corresponding to an object C is always projective. Furthermore, it has the categorical property which is the analogue of the property of being finitely generated in the category of modules over a ring.

Corollary 2. Every representable functor $\mathcal{C}(\_\_, C)$ is projective in the category of contravariant functors from $\mathcal{C}$ into the category of abelian groups.

PROOF: Consider a diagram

$$\begin{array}{ccccc} & & \mathcal{C}(\_\_, C) & & \\ & & \big\downarrow \tau & & \\ F & \longrightarrow & G & \longrightarrow & 0\,. \end{array}$$

By the Yoneda Lemma, τ is determined by the element $\tau_C(1_C) \in G(C)$. Since $F(C) \to G(C)$ is an epimorphism, we can choose a pre-image $x \in F(C)$ for $\tau_C(1_C)$. By the Yoneda Lemma, x corresponds to a natural transformation $\sigma\colon \mathcal{C}(\_\_, C) \to F$, and σ makes the resulting diagram commute, showing that the functor $\mathcal{C}(\_\_, C)$ is projective. ☑

Corollary 3. Let F be a contravariant functor from $\mathcal{C}$ into the category of abelian groups $\mathcal{A}b$ such that there exists an epimorphism

$$\rho : \coprod_I F_i \to F$$

for some be a family of contravariant functors F_i from $\mathcal{C}$ into $\mathcal{A}b$. For some object $C \in \mathcal{C}(\_\_,)$, let

$$\tau\colon \mathcal{C}(\_\_, C) \to F$$

be a natural transformation. Then there exists a finite subject $J \subseteq I$ such that

$$\tau(\mathcal{C}(\_\_, C)) \subseteq \rho\left(\coprod_J F_i\right)\,.$$

PROOF: By the Yoneda Lemma, τ is determined by $\tau_C(1_C) \in F(C)$. The condition that $\tau(\mathcal{C}(\_\_, C)) \subseteq \rho\left(\coprod_I F_i\right)$ amounts to the fact that $\tau(1_C) = \rho_C(\sum x_i)$ where $\sum x_i \in \coprod F_i(C)$ But in this sum, only finitely many terms will be non-zero. Therefore we can choose as J the set of subscripts correponding to those non-zero terms. ☑

This corollary shows that epimorphic images of representable functors are analogous to finitely generated modules in the category of modules over a ring.

Corollary 4. If C is an indecomposable object in $\mathcal{C}$, then the functor $\mathcal{C}(\_, C)$ is indecomposable and has at most one maximal subfunctor.

PROOF: By the Yoneda Lemma, $\mathrm{Nat}(\,\mathcal{C}(\_, C),\ \mathcal{C}(\_, C)\,) \approx \mathcal{C}(C,C) = \mathrm{End}_{\mathcal{C}}(C)$, and by hypothesis on the category $\mathcal{C}$, this is a local ring. Therefore $\mathcal{C}(\_, C)$ is indecomposable. Now if F and G are distinct maximal subfunctors of $\mathcal{C}(\_, C)$, then the inclusions induce a natural transformation $F \oplus G \to \mathcal{C}(\_, C)$ whose image is strictly larger than F and G. Therefore $F \oplus G \to \mathcal{C}(\_, C)$ must be an epimorphism. Since $\mathcal{C}(\_, C)$ is a projective functor by Corollary 2, this natural transformation must split, contradicting the Krull-Schmidt Theorem. ☑

One should not assume that $\mathcal{C}(\_, C)$ has finite length in the category of contravariant additive functors from $\mathcal{C}$ into $\mathcal{A}b$. (With considerable effort, we will prove in Lemma H that this is true in the case where there is a bound on the lengths of the indecomposable objects in $\mathcal{C}$.) However Corollary 4 shows that if C is indecomposable, then $\mathcal{C}(\_, C)$ has a simple quotient (in module theory, this would often be called the "cosocle"), and that this simple functor is essentially unique.

Corollary 5. If C and C' are objects in $\mathcal{C}$, then $\mathcal{C}(\_, C)$ and $\mathcal{C}(\_, C')$ are naturally isomorphic if and only if $C \approx C'$.

Lemma 6. Jordan-Hölder Theorem for functors from $\mathcal{C}$ into $\mathcal{A}b$.

Lemma A. If a contravariant functor F from $\mathcal{C}$ into $\mathcal{A}b$ has finite length n (in the category of all contravariant functors from $\mathcal{C}$ into $\mathcal{A}b$), then there exist at most n indecomposable objects $X \in \mathcal{C}$ such that $F(X) \neq 0$.

PROOF: By induction on the length of F, it suffices to prove that if F is simple then there is a unique C such that $F(C) = 0$. Since $F \neq 0$, there exists an object $C \in \mathcal{C}(\_,)$ such that $F(C) \neq 0$. By the Yoneda Lemma, this corresponds to a natural transformation $\tau\colon \mathcal{C}(\_, C) \to F$, which is necessarily an epimorphism since F is simple. Now suppose that C_1 is another object such that $F(C_1) \neq 0$. Then we get $\tau_1\colon \mathcal{C}(\_, C_1) \to F$. Now by Corollary 2 and Corollary 4, $\mathcal{C}(\_, C)$ and $\mathcal{C}(\_, C_1)$ are indecomposable projective functors. But by a standard argument from the theory of artinian rings, there cannot be epimorphisms from two non-isomorphic indecomposable projective objects onto a simple object. The remainder of the proof will simply re-iterate this argument.

Since by Corollary 2 the functor $\mathcal{C}(\_, C)$ is projective, there exists a natural transformation $f_*\colon \mathcal{C}(\_, C) \to \mathcal{C}(\_, C_1)$ such that $\tau_1 f_* = \tau$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}\,\tau & \longrightarrow & \mathcal{C}(\_, C) & \xrightarrow{\tau} & F & \longrightarrow & 0 \\ & & & & \downarrow f_* & & \| & & \\ 0 & \longrightarrow & \mathrm{Ker}\,\tau_1 & \longrightarrow & \mathcal{C}(\_, C_1) & \xrightarrow{\tau_1} & F & \longrightarrow & 0 \end{array}$$

(By the Yoneda Lemma, f_* is in fact induced by a map $f\colon C \to C_1$ as the notation suggests.) Now since F is simple, $\operatorname{Ker}\tau_1$ is a maximal subfunctor of $\mathcal{C}(\_\_, C_1)$. It follows that f_* must be an epimorphism, since its image is not contained in $\operatorname{Ker}\tau_1$ (because $\tau_1 f_* = \tau \neq 0$). But by Corollary 2, $\mathcal{C}(\_\_, C_1)$ is projective, so f_* must split. But by Corollary 4, $\mathcal{C}(\_\_, C)$ is indecomposable. Thus f_* must be an isomorphism. But then by Corollary 5, C is isomorphic to C_1. This shows that C is unique (up to isomorphism). ☑

Corollary B. If for every **simple** object $S \in \mathcal{C}$, the functor $\mathcal{C}(\_\_, S)$ has finite length, then $\mathcal{C}$ has only finitely many indecomposable objects.

PROOF: By assumption, every object $C \in \mathcal{C}$ has finite length. Therefore there is an epimorphism from C onto some simple object S. In particular, there is a simple object S such that $\mathcal{C}(C, S) \neq 0$. But by Corollary A, if $\mathcal{C}(\_\_, S)$ has finite length, then there can be only finitely many indecomposable objects C for which $\mathcal{C}(C, S) \neq 0$. Since by assumption the category contains only finitely many simple objects, this implies that there are only a finite number of indecomposable modules in $\mathcal{C}$. ☑

Definition. Let F be a covariant [contravariant] additive functor from $\mathcal{C}$ into $\mathcal{A}b$. If A is an object in $\mathcal{C}$ and $x \in F(A)$, then the ordered pair (A, x) is called a **minimal element** for F if $x \neq 0$ and $F(h)(x) = 0$ whenever h is a non-monic morphism from A into some object B [resp. whenever h is a non-epic morphism from some object B into A]. (A, x) is called **universally minimal** for F if $F(h)(x) = 0$ whenever $h\colon A \to B$ and h is not **split monic** [resp. $h\colon B \to A$ and h is not **split epic**].

Note that if (A, x) is a minimal element for F, then A must necessarily be indecomposable.

Lemma C. Suppose that the category $\mathcal{C}$ satisfies the ascending chain condition on monomoprhisms between indecomposable objects, i. e. there does not exist an infinite sequence of morphisms

$$C_1 \to C_2 \to C_3 \to \cdots$$

where all the objects C_i are indecomposable and the morphisms are monomorphisms and not isomorphisms. Let F be a non-zero additive covariant functor from $\mathcal{C}$ into $\mathcal{A}b$. Then there exists a universally minimal element for F.

Likewise, if $\mathcal{C}$ satisfies the descending chain condition on epimorphisms between indecomposable objects, and F is a non-zero contravariant additive functor from $\mathcal{C}$ into $\mathcal{A}b$, then there exists a universally minimal element for F.

PROOF: For the covariant case, start with an object A_1 such that $F(A_1) \neq 0$ and let $x_1 \neq 0 \in A_1$. Consider those epimorphisms from $f\colon A_1 \to B$ such that $F(f)(x_1) \neq 0$. Since A_1 has finite length (by hypothesis on $\mathcal{C}$), we can choose such an f so that the kernel is as large as possible. Now replace the pair (A_1, x_1) by $(f(A_1), f(x_1))$. In this way, we obtain an object A_1 and x_1 such that that $F(g)(x_1) = 0$ whenever g is an proper epimorphism from A_1 to an object B. And in fact, $F(g)(x_1) = 0$ whenever $g\colon A_1 \to B$ is any morphism which is not a monomorphism, as we see by considering the factorization of g as an epimorphism followed by a monomorphism. I. e. (A_1, x_1) is a minimal element for F. As noted above, A_1 must be indecomposable.

Now consider those monomorphisms $g\colon A_1 \to A_2$ such that $(A_2, F(g)(x_1))$ is a minimal element for F. Since by assumption $\mathcal{C}$ satisfies the ascending chain condition on proper monomorphisms between indecomposable objects, there must exist a minimal element (A, x) such that whenever $g\colon A \to A'$ is a monomorphism but not an isomorphism, $(A', F(g)(x))$ is not minimal.

We now claim that (A, x) is universally minimal. We need to see that $F(g)(x) = 0$ for all morphisms $g\colon A \to A'$ except when g is split monic. By assumption, (A, x) is a minimal element for F, so $F(g)(x) = 0$ unless g is monic. Now suppose that $g\colon A \to A'$ is monic and $F(g)(x) \neq 0$. Then repeating the preceding construction for the pair $(A', F(g)(x))$ yields a monomorphism $h\colon A' \to A''$ such that $(A'', F(h)F(g)(x))$ is minimal. But by assumption, this is only possible if $hg\colon A \to A''$ is an isomorphism. Therefore g splits. This shows that $F(g)(x) = 0$ for all $g\colon A \to A'$ unless g is a split monomorphism. I. e. (A, x) is a universally minimal element for F.

The proof of the contravariant case is analogous. ☑

Note that if there is a bound on the lengths of indecomposable objects in $\mathcal{C}$, then $\mathcal{C}$ certainly satisfies the ascending and descending chain conditions in the hypothesis of the preceding lemma.

Lemma D. Suppose the hypotheses of Lemma C. Then if C is an indecomposable object in $\mathcal{C}$, there exists a unique **simple** functor S which is a quotient of the functor $\mathcal{C}(\_\_, C)$, and there exists an exact sequence of functors

$$\mathcal{C}(\_\_, B) \to \mathcal{C}(\_\_, C) \to S \to 0\,.$$

PROOF: We claim that if G is an additive subfunctor of $\mathcal{C}(\_\_, C)$ such that for some object X, there exists a split epimorphism $f \in G(X) \subseteq \mathcal{C}(X, C)$, then $G = \mathcal{C}(\_\_, C)$. In fact, if

$$s\colon C \to X$$

is such that $fs = 1_C$, then for every object Y in $\mathcal{C}$ and every $h \in \mathcal{C}(Y, C)$,

$$h = fs\,h = (s\,h)^*(f) \in (s\,h)^*(G(X)) \subseteq G(Y)\,.$$

Thus $G(Y) = \mathcal{C}(Y, C)$ for every object Y, showing that $G = \mathcal{C}(\_\_, C)$.

Note further that since C is indecomposable, by the hypotheses on the category $\mathcal{C}$, $\operatorname{End}_{\mathcal{C}} C$ is a local ring. If follows that this that for any object X, the set of morphisms $h \in \mathcal{C}(X, C)$ which are not split epimorphisms is a subgroup of $\mathcal{C}(\_\_, C)$. We now see that if we define

$$F(X) = \{h \in \mathcal{C}(X, C) \mid h \text{ is } \textit{not} \text{ a split epimorphism}\}$$

for every object X, then F is the unique maximal subfunctor of $\mathcal{C}(\_\_, C)$.

Now if C is projective as well as indecomposable, then C has a unique maximal subobject B (see the proof of Corollary 4) and a morphism $h\colon X \to C$ is an epimorphism, necessarily split, if and only if $h(X) \nsubseteq B$. Thus if F is the functor defined above, $F = \mathcal{C}(\_\_, B)$, and there is an exact sequence

$$0 \to \mathcal{C}(\_\_, B) \to \mathcal{C}(\_\_, C) \to S \to 0$$

where S is the simple functor $\mathcal{C}(\_\_, C)/F$.

On the other hand, if the indecomposable object C is not projective, then $\mathrm{Ext}^1_{\mathcal{C}}(C, \_\_) \neq 0$ and by Lemma C there is a universally minimal element (A, x) for $\mathrm{Ext}^1_{\mathcal{C}}(C, \_\_)$. Let $x \in \mathrm{Ext}^1_{\mathcal{C}}(C, A)$ be represented by the short exact sequence

$$x: \quad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0\,.$$

Then $g_*\colon \mathcal{C}(\_\_, B) \to \mathcal{C}(\_\_, C)$. We claim that the image of g_* is the maximal subfunctor

$$F(X) = \{h \in \mathcal{C}(X, C) \mid h \text{ is } \textit{not} \text{ a split epimorphism}\}$$

of $\mathcal{C}(\_\_, C)$ defined above. In fact, $g_*(\mathcal{C}(\_\_, B)) \neq \mathcal{C}(\_\_, C)$ since $1_C \notin g_*(\mathcal{C}(C, B))$. (The identity map 1_C does not factor through g, since g is not a split epimorphism.) On the other hand, if $h \in \mathcal{C}(X, C)$ is not a split epimorphism, then the map $t\colon X \oplus B \to C$ induced by h and g cannot be split epic, because of the fact that $\mathrm{End}_{\mathcal{C}}\, C$ is a local ring. Now the inclusion $B \to X \oplus B$ induces a commutative diagram

$$\begin{array}{lccccccccc}
x\colon & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
& & & {\scriptstyle u}\big\downarrow & & \big\downarrow & & \big\| & & \\
u^*(x)\colon & 0 & \longrightarrow & \mathrm{Ker}\, t & \longrightarrow & X \oplus B & \xrightarrow{t} & C & \longrightarrow & 0\,.
\end{array}$$

The fact that t is not split means that $u_*(x) \neq 0$. But by assumption, (A, x) is universally minimal. Therefore u must be split monic. An easy diagram chase then shows that h factors through g, i. e. $h \in g_*(\mathcal{C}(X, B)) = g_*(\mathcal{C}(\_\_, B))(X)$. This shows that for every object X, $g_*(\mathcal{C}(X, B))$ consists of all morphisms from X to C which are not split epic, so that $g_*(\mathcal{C}(\_\_, B))$ is the unique maximal subfunctor of $\mathcal{C}(\_\_, C)$. ☑

The short exact sequence

$$x: \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

constructed in the proof of Lemma D and characterized by the fact that it is not split but that $u_*(x)$ is split exact for every morphism $u\colon A \to Y$ which is not split monic, will be one of the cornerstones of Auslander's later work with Idun Reiten. Auslander calls a sequence of this form an **almost split sequence**. (C. f. Maurice Auslander, *Representation theory of artin algebras III — almost split sequences*, Communications in Algebra **3**(1975), 239-94.)

Lemma E. (1) If a contravariant functor F from $\mathcal{C}$ into $\mathcal{A}b$ has a universally minimal element, then F has a simple subfunctor.

(2) If the hypotheses of Lemma C are valid for $\mathcal{C}$, then for every simple functor F there is an exact sequence of natural transformations

$$\mathcal{C}(\_\_, B) \longrightarrow \mathcal{C}(\_\_, A) \longrightarrow F \to 0$$

where A and B are objects in $\mathcal{C}$.

PROOF: (1) Let (A, x) be a universally minimal element for F. Recall that A is necessarily indecomposable. By the Yoneda Lemma, x corresponds to a natural transformation $\tau\colon \mathcal{C}(\_\_, A) \to F$, where for $f \in \mathcal{C}(X, A)$, $\tau_X(f) = F(f)(x)$. Now since (A, x) is universally minimal, if $g \in \mathcal{C}(X, A)$, then $0 = \tau_X(g) = F(g)(x)$ if and only if g is not split epic, i. e. if and only if

$g \notin G(X)$, where G is the unique maximal subfunctor of $\mathcal{C}(\_\_, A)$, as constructed in the proof of Lemma D. (Recall that since (A, x) is minimal, A is necessarily indecomposable.) Thus the sequence

$$0 \longrightarrow G \longrightarrow \mathcal{C}(\_\_, A) \xrightarrow{\ \tau\ } F$$

is exact. Since G is a maximal subfunctor of $\mathcal{C}(\_\_, A)$, the image of τ is thus a simple functor.

(2) Under the hypotheses of Lemma C, there exists a universally minimal element for F, and thus the preceding paragraph applies. But if F is simple, then the natural transformation $\tau \colon \mathcal{C}(\_\_, A) \to F$ must be epic. Therefore by Lemma D there is an exact sequence

$$0 \longrightarrow \mathcal{C}(\_\_, B) \longrightarrow \mathcal{C}(\_\_, A) \xrightarrow{\ \tau\ } F \longrightarrow 0\,. \quad \boxed{\checkmark}$$

Corollary F. Suppose the hypotheses of Lemma C hold. Then for every finite length functor F, there is a short exact sequence

$$0 \longrightarrow G \longrightarrow \mathcal{C}(\_\_, A) \xrightarrow{\ \tau\ } F \longrightarrow 0\,.$$

PROOF: Lemma E gives the result for simple functors. Now use induction on the length of F. $\boxed{\checkmark}$

Lemma G. Suppose that the hypotheses of Lemma C hold. Then every representatable contravariant functor $\mathcal{C}(\_\_, C)$ is generated by its finite length subfunctors. I. e. if G is a subfunctor of $\mathcal{C}(\_\_, C)$ such that every subfunctor of $\mathcal{C}(\_\_, C)$ having finite length is a subfunctor of G, then $G = \mathcal{C}(\_\_, C)$.

PROOF: Let G be the subfunctor of $\mathcal{C}(\_\_, C)$ generated by all its finite length subfunctors and suppose by way of contradiction that G is a proper subfunctor of $\mathcal{C}(\_\_, C)$. We will then show that there exists a subfunctor of $\mathcal{C}(\_\_, C)$ having finite length and not contained in G, a contradiction.

Let $F = \mathcal{C}(\_\_, C)/G$, so that there is a short exact sequence

$$0 \to G \to \mathcal{C}(\_\_, C) \to F \to 0.$$

By Lemma E, F has a simple subfunctor S and there is an exact sequence

$$\mathcal{C}(\_\_, B) \xrightarrow{\ g_*\ } \mathcal{C}(\_\_, A) \xrightarrow{\ \sigma\ } S \longrightarrow 0$$

Let G' be the inverse image of S in $\mathcal{C}(\_\_, C)$. Thus $G \subseteq G'$ and $G'/G = S \subseteq F$, yielding the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathcal{C}(__, B) & \xrightarrow{\ g_*\ } & \mathcal{C}(__, A) & \xrightarrow{\ \sigma\ } & S & \longrightarrow & 0 \\
 & & & & \rho\big\downarrow & & \big\Vert & & \\
0 & \longrightarrow & G & \xrightarrow{\ \subseteq\ } & G' & \xrightarrow{\ \tau\ } & S & \longrightarrow & 0 \\
 & & & & \big\downarrow\subseteq & & \big\downarrow\subseteq & & \\
0 & \longrightarrow & G & \xrightarrow{\ \subseteq\ } & \mathcal{C}(__, C) & \longrightarrow & F & \longrightarrow & 0\,.
\end{array}$$

The natural transformation $\rho\colon \mathcal{C}(\_\_, A) \to G'$ in this diagram exists because $\mathcal{C}(\_\_, A)$ is projective. Now let $G'' = \rho(\mathcal{C}(\_\_, A))$. We now have

$$\begin{array}{ccccccccc}
 & & \mathcal{C}(__, B) & \xrightarrow{g_*} & \mathcal{C}(__, A) & \xrightarrow{\sigma} & S & \longrightarrow & 0\\
 & & \big\downarrow & & {\scriptstyle \rho'}\big\downarrow & & \| & & \\
0 & \longrightarrow & \operatorname{Ker}\tau' & \longrightarrow & G'' & \xrightarrow{\tau'} & S & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle\subseteq} & & \big\downarrow{\scriptstyle\subseteq} & & \| & & \\
0 & \longrightarrow & G & \xrightarrow{\subseteq} & G' & \longrightarrow & S & \longrightarrow & 0\,.
\end{array}$$

The point of this construction is that G'', being the image of the representable functor $\mathcal{C}(\_\_, A)$, is in some sense reasonably small (Auslander would call it finitely generated), and yet $\tau(G'') = S$, so in particular $G'' \nsubseteq \operatorname{Ker}\tau = G$. We will show that G'' has finite length.

Since S is simple, it suffices to prove that $\operatorname{Ker}\tau'$ has finite length. Note that $\operatorname{Ker}\tau' = \rho g_*(\mathcal{C}(\_\_, B))$. By assumption, G is generated by its finite length subfunctors. This means that there is a natural epimorphism $\varepsilon\colon \coprod_I F_i \to G$ where each functor F_i has finite length. Thus we have

$$\begin{array}{ccccc}
 & & \mathcal{C}(__, B) & & \\
 & & \big\downarrow{\scriptstyle \rho g_*} & & \\
\coprod_I F_i & \xrightarrow{\varepsilon} & G & \longrightarrow & 0\,.
\end{array}$$

Now apply Corollary 3 to see that the index set I here can be replaced by a finite set J such that

$$\operatorname{Ker}\tau' = \rho g_*(\mathcal{C}(\_\_, B)) \subseteq \varepsilon\left(\coprod_J F_i\right).$$

Then $\varepsilon\left(\coprod_J F_i\right)$ has finite length and it follows that $\operatorname{Ker}\tau'$ has finite length. ☑

Lemma H. Suppose that the hypotheses of Lemma C hold. Then every representable contravariant additive functor $\mathcal{C}(\_\_, C)$ has finite length.

PROOF: The reasoning duplicates the last paragraph of the proof of Lemma G. Namely, by Lemma G, $\mathcal{C}(\_\_, C)$ is generated by its subfunctors with finite length. Thus there is a natural epimorphism

$$\coprod_I F_i \to \mathcal{C}(\_\_, C)$$

where each F_i is a subfunctor of $\mathcal{C}(\_\_, C)$ with finite length. But by Corollary 3, we may replace I by a finite index set. It follows that $\mathcal{C}(\_\_, C)$ has finite length. ☑

Proof of the Main Theorem. It now follows from Corollary B that if the hypotheses of Lemma C hold, then $\mathcal{C}$ has (up to isomorphism) only finitely many indecomposable objects. ☑

Another Theorem

We now give the proof by Auslander of another theorem by Roiter.

Theorem. Let Λ be an artinian ring such that there are (up to isomorphism) only finitely many finitely generated indecomposable Λ-modules. Then every Λ-module is a direct sum of finitely generated Λ-modules.

PROOF: (1) If I is an indecomposable injective Λ-module then the submodules of I are indecomposable (because the 0 submodule of I is irreducible), so there is a bound on the lengths of the finitely generated submodules of I. Thus there is a maximal finitely generated submodule of I. But clearly this submodule must be I itself. Thus every indecomposable injective Λ-module is finitely generated.

(2) Let G be any generator for the category of left Λ-modules and let $\Gamma = \operatorname{End}_\Lambda G$. Then $\operatorname{Hom}_\Lambda(G, \_\,)$ is a fully faithful functor from the category of left Λ-modules to the category of right Γ-modules. We claim that if $P = \operatorname{Hom}_\Lambda(G, \Lambda)$, then the functor $\operatorname{Hom}_\Gamma(P, \_\,)$ is a left inverse for $\operatorname{Hom}_\Lambda(G, \_\,)$. In fact, since G is a generator for the category of left Λ-modules, there is a surjection $\pi\colon \bigoplus_1^n G \to \Lambda$, which is split because Λ is a free Λ-module. Let $\sigma\colon \Lambda \to \bigoplus G$ be a splitting for π and let σ_i and π_i be the components of σ and π and $g_i = \sigma_i(1)$. In other words, $\sigma(1) = (g_1, \ldots, g_n)$, $\pi_i\colon G \to \Lambda$, and $\sum_1^n \pi_i(g_i) = 1$. Note also that $\pi_i \in P = \operatorname{Hom}_\Lambda(G, \Lambda)$. Define a natural transformation

$$\theta\colon \operatorname{Hom}_\Gamma(P, \operatorname{Hom}_\Lambda(G, \_\,)\,) \to \_$$

by defining

$$\theta(\varphi) = \sum_1^n \varphi(\pi_i)(g_i)\,.$$

(This makes sense because $\pi_i \in P$ and if $\varphi\colon P \to \operatorname{Hom}_\Lambda(G, M)$ then $\theta_M(\varphi) = \sum \varphi(\pi_i)(g_i) \in M$.) Now for a given left Λ-module M and $m \in M$, let $\mu \in \operatorname{Hom}_\Lambda(\Lambda, M)$ be the map such that $\mu(1) = m$. Then $\mu_* \in \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(G, \Lambda), \operatorname{Hom}_\Lambda(G, M)\,) = \operatorname{Hom}_\Gamma(P, \operatorname{Hom}_\Lambda(G, M))$ and

$$\theta_M(\mu_*) = \sum \mu_*(\pi_i)(g_i) = \sum \mu\pi_i\sigma_i(1) = \mu(1) = m\,.$$

Thus $\theta_M\colon \operatorname{Hom}_\Gamma(P, \operatorname{Hom}_\Lambda(G, M)\,) \to M$ is an epimorphism. We will now show that it is monic. Let $\alpha \in \operatorname{Ker}\theta_M \subseteq \operatorname{Hom}_\Gamma(P, \operatorname{Hom}_\Lambda(G, M))$. Then

$$0 = \theta_M(\alpha) = \sum \alpha(\pi_i)(g_i) = \sum \alpha(\pi_i)(\sigma_i(1)) \in M\,.$$

We claim that $\alpha = 0$. To show this, we must show that $\alpha(\beta) = 0$ for every $\beta \in P = \operatorname{Hom}_\Lambda(G, \Lambda)$. In fact, let $g' \in G$. Note that $\sigma_i\beta \in \Gamma = \operatorname{End}_\Lambda G$ and $\alpha(\pi_i) \in \operatorname{Hom}_\Lambda(G, M)$ and remember that α is Γ-linear **on the right**. Then

$$\begin{aligned}\alpha(\beta)(g') &= \sum \alpha(\pi_i\sigma_i\beta)(g') = \sum \alpha(\pi_i)\sigma_i\beta(g') \\ &= \sum \beta(g')\alpha(\pi_i)\sigma_i(1) = \beta(g')\theta_M(\alpha) = 0,\end{aligned}$$

using the fact that $\beta(g') \in \Lambda$ and σ_i and $\alpha(\pi_i)$ are Λ-linear. Since this is true for all $\beta \in P$, we see that $\alpha = 0$, showing that θ is monic and thus an isomorphism.

(3) Now suppose further that G is finitely generated. Since Λ is a summand of a finite direct sum of copies of G, $P = \mathrm{Hom}_\Lambda(G, \Lambda)$ is a finitely generated projective right Γ-module. On the other hand, by step (2), $\mathrm{Hom}_\Gamma(P, \Gamma) = G$. Since both $\mathrm{Hom}_\Lambda(G, \_)$ and $\mathrm{Hom}_\Gamma(P, \_)$ commute with coproducts (because P and G are finitely generated), they thus take summands of coproducts of copies of G to Γ-projective modules and vice-versa.

(4) By hypothesis there are, up to isomorphism, only finitely many finitely generated indecomposable Λ-modules. Now choose the finitely generated generator G so that every finitely generated indecomposable Λ-module is isomorphic to a summand of G. With this assumption, if M is a finitely generated Λ-module then M is isomorphic to a summand of G^r for some finite r and the Γ-module $\mathrm{Hom}_\Lambda(G, M)$, is projective. We now claim that Γ has global dimension at most 2. It suffices to prove that if N is finitely generated and $P_1 \to P_0 \to N \to 0$ is exact, then $\mathrm{Ker}(P_1 \to P_0)$ is projective. Suppose first that P_1 and P_0 are finitely generated projective Λ-modules. By steps (2) and (3), $P_i \approx \mathrm{Hom}_\Lambda(G, M_i)$ for some finitely generated Λ-modules M_i, and any morphism $P_1 \to P_0$ is induced by a map $\delta\colon M_1 \to M_0$. Let $M_3 = \mathrm{Ker}\,\delta$. Then we get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(G, M_3) \to P_1 \to P_0\,,$$

showing that the kernel of the map $P_1 \to P_0$ is projective when P_1 and P_0 are both finitely generated projective Γ-modules.

(5) Now it was shown in the proof of the previous theorem that the functor $\mathrm{Hom}_\Lambda(\_, G)$ has finite length. It then follows easily that $\Gamma = \mathrm{End}_\Lambda G$ is right artinian. (In most cases, $\mathrm{End}_\Lambda G$ would be an artinian ring in any case for any finitely generated Λ-module G.) Hence all finitely generated right Γ-modules are finitely presented. This completes the proof that Γ has global dimension at most 2.

(6) Now let M be any Λ-module, not necessarily finitely generated, and form an injective resolution

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1$$

By step (1), I_0 and I_1 are direct sums of finitely generated indecomposable Λ-modules, hence, by the assumption on G, are isomorphic to direct summands of coproducts of copies of G. Thus in the sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(G, M) \longrightarrow \mathrm{Hom}_\Lambda(G, I_0) \longrightarrow \mathrm{Hom}_\Lambda(G, I_1)$$

the second and third Γ-modules are projective. Since Γ has global dimension at most 2, $\mathrm{Hom}_\Lambda(G, M)$ must also be a projective Γ-module. But by Step (2),

$$M \approx \mathrm{Hom}_\Gamma(P, \mathrm{Hom}_\Lambda(G, M))\,.$$

Thus by step (3) M is a coproduct of direct summands of G, hence a coproduct of finitely generated modules. ☑