A COURSE IN HOMOLOGICAL ALGEBRA
CHAPTER **: GORENSTEIN RINGS & MODULES
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References: Kaplansky, Commutative Rings (First Edition), Chapter 4, Sections 4 through 6.

H. Bass, The Ubiquity of Gorenstein Rings, Math Zeitschrift **82** (1963) 8–28.

Sharp, Math Zeitschrift **112**(1969), 340–56,   Math Zeitschrift **115**(1970), 117-39.

Important: Kaplansky, Commutative Rings (Second Edition), Chapter 4, Sections 5 and 6, pp. 156–168.

The second edition of Kaplansky contains proofs for some of these results simpler than those given here. Unfortunately, Kaplansky uses the word "grade" to mean depth, conflicting with Bass's more standard use of that word (as indicated below).

Assume throughout that R is a commutative noetherian ring. $E(M)$ denotes the injective envelope of an R-module M.

Lemma. Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let E be the injective envelope of $R/\mathfrak{m}$. Then the functor $\operatorname{Hom}_R(\_\,, E)$ restricted to the category of finite-length R-modules preserves length. (Kaplansky, Second Edition, Theorem 220.) Conversely, if M is an R-module such that $\operatorname{Hom}_R(M, E)$ has finite length, then M has finite length.

PROOF: Consider first a module S with length 1 (i. e. a simple module). Then $S \approx R/\mathfrak{m}$ (so E is isomorphic to the injective envelope of S). Now if $\varphi \in \operatorname{Hom}_R(S, E)$, then $\varphi(S)$ is simple (unless trivial) and $\varphi(S) \cap (R/\mathfrak{m}) \neq 0$ since E is an essential extension of $R/\mathfrak{m}$. Thus $\varphi(S) = R/\mathfrak{m}$, since both these modules are simple. From this we see that $\operatorname{Hom}_R(S, E) \approx \operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{m}) \approx R/\mathfrak{m}$, so that $\operatorname{length} \operatorname{Hom}_R(S, E) = 1$. Since the functor $\operatorname{Hom}_R(\_\,, E)$ preserves (or, more precisely, reverses) short exact sequences, it now follows by induction that for all R-modules M with finite length, $\operatorname{length} \operatorname{Hom}_R(M, E) = \operatorname{length} M$.

Now suppose that M is an R-module such that $\operatorname{Hom}_R(M, E)$ has finite length. Consider first the case where M is finitely generated. M contains submodules N such that M/N has finite length (for instance, $N = M$). By the preceding, for such an N, $\operatorname{Hom}_R(M/N, E)$ has the same length as M/N. But the surjection $M \to M/N$ induces a monomorphism $\operatorname{Hom}_R(M/N, E) \to \operatorname{Hom}_R(M, E)$, so $\operatorname{length} M/N \leq \operatorname{length} \operatorname{Hom}_R(M, E)$. Thus there is a bound on the length of M/N for those N such that M/N has finite length. But if M/N has finite length and $N \neq 0$, then since N is noetherian, it contains a maximal proper submodule N' and $\operatorname{length} M/N' = 1 + \operatorname{length} M/N$. Therefore eventually we must have $N = 0$, so that M has finite length.

If M is not finitely generated and $\mathrm{Hom}_R(M, E)$ has finite length, consider a finitely generated submodule M' of M. The inclusion map from M' to M induces a surjection $\mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M', E)$ (because E is injective). Therefore $\mathrm{Hom}_R(M', E)$ has finite length and so M' has finite length by the preceding paragraph, and in fact the lengths of the finitely generated submodules of M are all bounded by $\mathrm{length}\,\mathrm{Hom}_R(M, E)$. Furthermore, if M'' is also finitely generated and $M' \subsetneq M''$ then $\mathrm{length}\,M'' > \mathrm{length}\,M'$. It follows that there exists a maximally finitely generated submodule of M. But clearly this is only possible if this submodule is M itself. Therefore M is finitely generated and hence, by the preceding paragraph, has finite length. ☑

Theorem. Let R be a commutative noetherian local ring and let $\mathfrak{m}$ be its unique maximal prime ideal. The following conditions are equivalent:

(1) R is artinian.

(2) R has finite length (as an R-module).

(3) $\mathfrak{m}$ is the only prime ideal in R.

(4) $\mathrm{Ass}\,R = \{\mathfrak{m}\}$.

(5) $\mathfrak{m}^k = 0$ for some positive integer k.

(6) The injective envelope E of $R/\mathfrak{m}$ is finite generated.

(7) There exists a finitely generated injective R-module.

PROOF: (2) $\Rightarrow$ (1): Clear.

(1) $\Rightarrow$ (4): If R is artinian and $\mathfrak{p} \in \mathrm{Ass}\,R$, then R contains a copy of $R/\mathfrak{p}$. Therefore $R/\mathfrak{p}$ is artinian. But $R/\mathfrak{p}$ is an integral domain, and an artinian integral domain is a field. Therefore $\mathfrak{p}$ is a maximal ideal, so $\mathfrak{p} = \mathfrak{m}$.

(4) $\Rightarrow$ (5): If $\mathrm{Ass}\,M = \{\mathfrak{m}\}$ then for some k, $\mathfrak{m}^k 1 = 0 \in R$ so $\mathfrak{m}^k = 0$.

(5) $\Rightarrow$ (2): For each i from 0 to k, $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is finitely generated and a vector space over the field $R/\mathfrak{m}$, hence is a finite-length R-module. Thus by induction, $R/\mathfrak{m}^i$ has finite length. If $\mathfrak{m}^k = 0$, this shows that R has finite length.

(5) $\Rightarrow$ (3): If $\mathfrak{m}^k = 0$ and $\mathfrak{p}$ is a prime ideal, then $\mathfrak{m}^k \subseteq \mathfrak{p}$ so $\mathfrak{m} \subseteq \mathfrak{p}$. Since $\mathfrak{m}$ is maximal, $\mathfrak{p} = \mathfrak{m}$.

(3) $\Rightarrow$ (4): Clear since $\mathrm{Ass}\,R \neq \{\varnothing\}$.

(2) $\Leftrightarrow$ (6): If $E = E(R/\mathfrak{m})$ is finitely generated, then it has finite length, since all its associated primes are maximal. And conversely, if it has finite length then it is certainly finitely generated. But $E \approx \mathrm{Hom}_R(R, E)$. Therefore the preceding lemma implies that E has finite length if and only if R has finite length.

(6) $\Rightarrow$ (7): Clear.

(7) $\Rightarrow$ (6): Let M be a non-trivial finitely generated injective module and let $\mathfrak{p} \in \mathrm{Ass}\,M$. Then M contains a copy of $R/\mathfrak{p}$. Since $R/\mathfrak{p}$ is an essential submodule of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, M therefore contains a submodule isomorphic to $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Therefore $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is finitely generated. Now let $x \notin \mathfrak{p}$. Then

multiplication by x is an isomorphism on $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. If $x \in \mathfrak{m}$, this contradicts Nakayama's Lemma. Therefore $x \notin \mathfrak{m}$ and we conclude that $\mathfrak{p} = \mathfrak{m}$. This shows that $\mathfrak{m} \in \operatorname{Ass} M$, and so M contains a submodule isomorphic to $R/\mathfrak{m}$. Since M is injective, it thus contains a copy of $E = E(R/\mathfrak{m})$. Therefore E is finitely generated.

$(6) \Rightarrow (4)$ [Alternate proof]: Since $\operatorname{Ass} E = \{\mathfrak{m}\}$, it follows that for each $e \in E$, $\mathfrak{m}^k e = 0$ for some k. Thus if E is finitely generated, then $\mathfrak{m}^k E = 0$ for some k. It now suffices to show that E is faithful in order to conclude that $\mathfrak{m}^k = 0$. Suppose in fact, by way of contradiction, that $rE = 0$ for some $r \neq 0 \in R$. The principal ideal (r) is isomorphic to $R/\mathfrak{a}$, where $\mathfrak{a} = \operatorname{ann} r$. Since $\mathfrak{a} \subseteq \mathfrak{m}$, there is a surjection $(r) \to R/\mathfrak{m}$ (so $\varphi(r) \neq 0$). Since $E = E(R/\mathfrak{m})$, this extends to a map $\varphi \colon R \to E$. Then $\varphi(r) = r\varphi(1) = 0$ since $rE = 0$. This is a contradiction, showing that no such r exists and therefore E is faithful. Since $\mathfrak{m}^k E = 0$, it follows that $\mathfrak{m}^k = 0$. $\boxed{\checkmark}$

Definition. An submodule N of an R-module M is called **irreducible** if it is not the intersection of two submodules which properly contain it.

Recall the following characterization of indecomposable injective modules.

Lemma. If M is an injective module over a commutative noetherian ring R, the following conditions are equivalent:

(1) M is indecomposable.

(2) M is the injective envelope of every non-trivial submodule.

(3) The zero submodule is irreducible in M.

(4) For some prime ideal $\mathfrak{p}$, $\operatorname{Ass} M = \{\mathfrak{p}\}$ and M is isomorphic to the injective envelope of $\mathfrak{p}$,

Proposition. If E is an injective module, then E has a direct summand isomorphic to $R/\mathfrak{p}$ if and only if $\mathfrak{p} \in \operatorname{Ass} E$.

PROOF: $\operatorname{Ass} E$ includes $\mathfrak{p}$ if and only if E contains a submodule isomorphic to $R/\mathfrak{p}$. If this is the case, the it will also contain a submodule isomorphic to the injective envelope of E, which will necessarily be a direct summand since it is injective. $\boxed{\checkmark}$

Corollary. A local ring R is injective as an R-module if and only if R is artinian and the zero ideal is irreducible in R. Furthermore, in this case, $R \approx E(R/\mathfrak{m})$.

PROOF: Since R is a finitely generated R-module, if it is injective then R is artinian by the Theorem above and $\operatorname{Ass} R = \{\mathfrak{m}\}$. Thus R contains a submodule (ideal) isomorphic to $R/\mathfrak{m}$, and so if R is injective, it contains a submodule isomorphic to $E = E(R/\mathfrak{m})$. But $E \approx \operatorname{Hom}_R(R, E)$, so by a lemma above, E and R have the same length. Thus if R is injective, it is isomorphic to $E(R/\mathfrak{m})$. Then by the lemma above, the zero ideal is irreducible in R.

Conversely, if R is artinian then it contains a minimal non-zero ideal I, which must be isomorphic to $R/\mathfrak{m}$. If moreover the zero ideal in R is irreducible, then R is an essential extension of I, so R is

isomorphic to a submodule of $E(I) \approx E(R/\mathfrak{m})$. But, as in the previous paragraph, we see that $E(R/\mathfrak{m})$ and R have the same length. Thus $R = E(I)$, hence R is injective. ☑

Definition. If $\mathfrak{p}$ is a prime ideal in R, we write $k(\mathfrak{p})$ to denote $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. (This is the quotient field of $R/\mathfrak{p}$.)

Definition. A **minimal injective resolution** of a module M is a complex

$$0 \xrightarrow{d_{-1}} M \longrightarrow E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \ldots$$

such that each E_i is the injective envelope of $d_{i-1}(E_{i-1})$.

Minimal injective resolutions are unique up to isomorphism. Since R is noetherian, each injective module E_i is a direct sum (often infinite) of indecomposable injective modules. By the preceding lemma, each such indecomposable summand is isomorphic to $E(R/\mathfrak{p})$, for some prime ideal $\mathfrak{p}$.

Notation. If in a given decomposition of E_i as a direct sum of indecomposable modules, k of these indecomposable modules are isomorphis to $E(R/\mathfrak{p})$, we write $\mu_i(\mathfrak{p}, M) = k$. We also write $\mu_i(\mathfrak{p})$ to denote $\mu_i(\mathfrak{p}, R)$.

It is easy to see that $\mu_i(\mathfrak{p}, M)$ is the largest integer k such that E_i contains a submodule isomorphic to $(R/\mathfrak{p})^k$. In particular, we have the following.

Note. $\mu_0(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$.

Proposition. The injective dimension of an R-module M is the smallest positive integer n such that $\operatorname{Ext}_R^{n+1}(R/\mathfrak{a}, M) = 0$ for every ideal $\mathfrak{a}$ of R.

Lemma. For any multiplicative set S, $S^{-1}E(M)$ is the injective envelope of $S^{-1}M$ as an $S^{-1}R$-module.

Corollary. Minimal injective resolutions localize. Therefore inj. $\dim_{S^{-1}R} S^{-1}M \leq$ inj. $\dim_R M$ and inj. $\dim M$ is the supremum of inj. $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ over all maximal ideals $\mathfrak{p}$ of R.

Corollary. If $\mathfrak{p}$ does not blow up in in $S^{-1}R$ then $\mu_i(S^{-1}\mathfrak{p}, S^{-1}M) = \mu_i(\mathfrak{p}, M)$.

Lemma. $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_R(R/\mathfrak{p}, M)_{\mathfrak{p}}$.

PROOF: Localizing, we may suppose that $\mathfrak{p}$ is maximal. Then $k(\mathfrak{p}) = R/\mathfrak{p}$ is a simple module, and $\operatorname{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{q})) = 0$ for $\mathfrak{q} \neq \mathfrak{p}$, since $\mathfrak{p} \notin \operatorname{Ass} R/\mathfrak{q} = \{\mathfrak{q}\}$. Furthermore $\operatorname{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{p})) \approx \operatorname{Hom}_R(R/\mathfrak{p}, R/\mathfrak{p}) \approx R/\mathfrak{p}$, since $E(R/\mathfrak{p})$ is an essential extension of the simple module $R/\mathfrak{p}$. It then follows that $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Hom}_R(R/\mathfrak{p}, E_i) = \dim_{k(\mathfrak{p})} \operatorname{Hom}_R(k(\mathfrak{p}), E_i)$.

Now if $\varphi \neq 0 \in \operatorname{Hom}_R(k(\mathfrak{p}), E_i)$, then $\varphi(k(\mathfrak{p}))$ is a simple submodule of E_i and since E_i is an essential extension of $d_{i-1}(E_{i-1})$, it follows that $\varphi(k(\mathfrak{p})) \subseteq d_{i-1}(E_{i-1})$. Therefore $d_i\varphi(k(\mathfrak{p})) = 0$. This shows that the complex

$$\cdots \xrightarrow{\ d^*_{i-1}\ } \operatorname{Hom}_R(k(\mathfrak{p}), E_i) \xrightarrow{\ d^*_i\ } \operatorname{Hom}_R(k(\mathfrak{p}), E_{i+1}) \xrightarrow{\ d^*_{i+1}\ } \cdots$$

has trivial differentiation. Therefore $\operatorname{Ext}^i_R(k(\mathfrak{p}), M_{\mathfrak{p}}) \approx \operatorname{Hom}_R(k(\mathfrak{p}), E_i)$, and $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Hom}_R(k(\mathfrak{p}), E_i) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_R(k(\mathfrak{p}), M_{\mathfrak{p}})_{\mathfrak{p}}$. $\boxed{\checkmark}$

Corollary. If M is finitely generated, then $\mu_i(\mathfrak{p}, M) < \infty$.

Hauptlemma Let $\mathfrak{q} \subsetneq \mathfrak{p}$ be adjacent primes and let M be finitely generated. If $\mu_k(\mathfrak{q}, M) \neq 0$, then $\mu_{k+1}(\mathfrak{p}, M) \neq 0$. (In other words, if $\mathfrak{q} \in \operatorname{Ass} E_k$ then $\mathfrak{p} \in \operatorname{Ass} E_{k+1}$.)

PROOF: We may suppose that R is local with maximal ideal $\mathfrak{p}$. Let $k = R/\mathfrak{p}$.

(1) We claim that there exists a finite length R-module C such that $\operatorname{Ext}^{k+1}_R(C, M) \neq 0$. In fact, let $B = R/\mathfrak{q}$ and $C = B/xB = R/(\mathfrak{q}, x)$, where x is an element of $\mathfrak{p}$ not belonging to $\mathfrak{q}$. Then x is regular on B, so we have an exact sequence

$$0 \longrightarrow B \xrightarrow{\ x\ } B \longrightarrow C \longrightarrow 0,$$

which induces a long exact sequence, part of which is

$$\cdots \longrightarrow \operatorname{Ext}^k_R(B, M) \xrightarrow{\ x\ } \operatorname{Ext}^k_R(B, M) \longrightarrow \operatorname{Ext}^{k+1}_R(C, M) \longrightarrow \cdots$$

Now $\operatorname{Ext}^k_R(B, M) \neq 0$ since $\mu_k(\mathfrak{q}, M) \neq 0$, and by Nakayama's Lemma, multiplication by x is not surjective, so $\operatorname{Ext}^{k+1}_R(C, M) \neq 0$.

Now among such C, choose one with smallest possible length. We easily see that in fact, $C = k$. Thus $\operatorname{Ext}^{k+1}_R(k, M) \neq 0$, so that $\mu_{k+1}(\mathfrak{p}, M) \neq 0$. $\boxed{\checkmark}$

Corollary. Let M be a finitely generated R-module with finite injective dimension r. Let $\mathfrak{p} \in \operatorname{Supp} M$. Then

(1) $\operatorname{height}_M \mathfrak{p} \leq \operatorname{inj.\ dim} M_{\mathfrak{p}}$.

(2) If $\mathfrak{p}$ is a prime ideal such that $\mu_r(\mathfrak{p}, M) \neq 0$, then $\mathfrak{p}$ is maximal.

PROOF: (1) If $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q}$ is minimal in $\operatorname{Supp} M$, then $\mathfrak{q} \in \operatorname{Ass} M_{\mathfrak{p}} \subseteq \operatorname{Ass} E(M_{\mathfrak{p}})$, so $\mu_0(\mathfrak{q}, M_{\mathfrak{p}}) \neq 0$. Now use induction and the Hauptlemma to see that if $\operatorname{height}_M \mathfrak{p} = n$ then $\mu_n(\mathfrak{p}, M_{\mathfrak{p}}) \neq 0$, so that $n \leq \operatorname{inj.\ dim} M_{\mathfrak{p}}$.

(2) Clear from the Hauptlemma. $\boxed{\checkmark}$

Remark. In particular, inj. $\dim R$ is at least as large as the Krull dimension of R.

Corollary. If R is local with maximal ideal $\mathfrak{m}$ and M is an R-module with finite injective dimension, then inj. $\dim M$ is the largest integer r such that $\mu_r(\mathfrak{m}, M) \neq 0$.

PROOF: Clearly $\mu_i(\mathfrak{m}, M) = 0$ for $i > $ inj. $\dim M$. And if $r = $ inj. $\dim M$, then $E_r \neq 0$ so $\mu_r(\mathfrak{p}, M) \neq 0$ for some prime $\mathfrak{p}$. Since R is local, by the preceding corollary $\mathfrak{p} = \mathfrak{m}$. $\boxed{\checkmark}$

At this point, we recall a theorem from the chapter on projective dimension.

Theorem. Let $(R, \mathfrak{m})$ be local and $x_1, \ldots, x_r$ a regular M-sequence. Then

$$\text{proj. } \dim M/(x_1, \ldots, x_r)M = r + \text{proj. } \dim M.$$

In particular, if I is an ideal generated by a regular R-sequence of length s, then proj. $\dim R/I = s$.

Corollary. Let R be local with maximal ideal $\mathfrak{m}$. Let M be a finitely generated R-module. Then either inj. $\dim M = \text{depth}_{\mathfrak{m}} R$ or inj. $\dim M = \infty$.

PROOF: Assume that M has finite injective dimension. Let I be an ideal generated by a maximal regular R-sequence in $\mathfrak{m}$. By the theorem above, $\text{depth}_{\mathfrak{m}} R = \text{depth}_I R = \text{proj. } \dim R/I$. Therefore it suffices to prove that inj. $\dim M = \text{proj. } \dim R/I$. Now since M is finitely generated, it has a maximal proper submodule M' and $M/M' \approx k$, so there is a surjection $M \to k$. Now if $s = \text{proj. } \dim R/I$, then we have seen previously that $\text{Ext}_R^s(R/I, k) \neq 0$. Furthermore, the functor $\text{Ext}_R^s(R/I, \_)$ is right exact. Therefore $\text{Ext}_R^s(R/I, M) \neq 0$ and $\text{depth}_{\mathfrak{m}} R = \text{proj. } \dim R/I$ can be characterized as the largest integer s such that $\text{Ext}_R^s(R/I, M) \neq 0$.

Now let $r = $ inj. $\dim M$. Then $\text{Ext}_R^r(\_, M)$ is right exact. Now $\mu_r(\mathfrak{p}, M) \neq 0$ for some prime ideal $\mathfrak{p}$, and from the Hauptlemma we see that $\mathfrak{p}$ must be maximal, i.e. $\mathfrak{p} = \mathfrak{m}$. This shows that that $\text{Ext}_R^r(k, M) \neq 0$ (see a lemma above). Now since I is generated by a maximal regular R-sequence in $\mathfrak{m}$, $\mathfrak{m}$ consists of zero divisors on R/I, so $\mathfrak{m} \in \text{Ass } R/I$, and therefore R/I contains a copy of $k = R/\mathfrak{m}$, and it follows from the right exactness of $\text{Ext}_R^r(\_, M)$ that $\text{Ext}_R^r(R/I, M) \neq 0$. From this it follows that inj. $\dim M$ can be characterized the largest integer r such that $\text{Ext}_R^r(R/I, M) \neq 0$. But comparing this with the characterization of proj. $\dim R/I$ in the paragraph above, we that inj. $\dim M = \text{proj. } \dim R/I = \text{depth}_{\mathfrak{m}} R$. $\boxed{\checkmark}$

We now recall two more theorems from the chapter on projective dimension.

Theorem. Let R be a noetherian ring and M a finitely generated R-module and $\mathfrak{a}$ an ideal such that $\mathfrak{a}M \neq M$. If r is the smallest integer such that $\text{Ext}_R^r(R/\mathfrak{a}, M) \neq 0$, then every regular M-sequence in $\mathfrak{a}$ has length r. Thus $\text{depth}_{\mathfrak{a}} M = r$.

Theorem. If $x_1, \ldots, x_r$ is a regular M-sequence in $\mathfrak{a}$ and $r = \text{depth}_{\mathfrak{a}} M$, then

$$\text{Ext}_R^r(R/\mathfrak{a}, M) \approx \text{Hom}_R(R/\mathfrak{a}, M/(x_1, \ldots, x_r)M).$$

Corollary. If $\mathfrak{p} \in \operatorname{Supp} M$, then $\operatorname{depth}_{\mathfrak{p}} M \leq \operatorname{depth}_{\mathfrak{p} R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$. If $\operatorname{depth}_{\mathfrak{q}} M > \operatorname{depth}_{\mathfrak{p}} M$ for every prime $\mathfrak{q}$ strictly containing $\mathfrak{p}$ (for instance if $\mathfrak{p} = \mathfrak{m}$), then $\operatorname{depth}_{\mathfrak{p}} M = \operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$.

PROOF: Since $\operatorname{Ext}^r_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}, M_{\mathfrak{p}}) \approx \operatorname{Ext}^r_R(R/\mathfrak{p}, M)_{\mathfrak{p}}$, it follows from the first of the two theorems above that $\operatorname{depth}_{\mathfrak{p}} M \leq \operatorname{depth}_{\mathfrak{p} R_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Now if $s = \operatorname{depth}_{\mathfrak{p}} M$ and I is an ideal generated by a maximal regular M-sequence in $\mathfrak{p}$, then by the second theorem above, $\operatorname{Ext}^s_R(R/\mathfrak{p}, M)_{\mathfrak{p}} = \operatorname{Hom}_R(R/\mathfrak{p}, M/IM)_{\mathfrak{p}}$. By the maximality of I, $\mathfrak{p}$ consists of zero divisors on M/IM, so that $\mathfrak{p}$ is contained in some prime $\mathfrak{q}$ in $\operatorname{Ass} M/IM$. If $\operatorname{depth}_{\mathfrak{q}} M > \operatorname{depth}_{\mathfrak{p}} M$ for every prime ideal $\mathfrak{q}$ strictly containing $\mathfrak{p}$, then $\mathfrak{q} \notin \operatorname{Ass} M/IM$ and it follows that $\mathfrak{p} \in \operatorname{Ass} M/IM$. Then there is a monomorphism $\varphi \colon R/\mathfrak{p} \to M/IM$. Then $\varphi_{\mathfrak{p}} \neq 0 \in \operatorname{Hom}(R/\mathfrak{p}, M/IM)_{\mathfrak{p}}$ (see the second theorem above), so that $\operatorname{Ext}^s_R(R_{\mathfrak{p}}, M)_{\mathfrak{p}} \neq 0$ and so $s = \operatorname{depth}_{\mathfrak{p}} M \geq \operatorname{depth}_{\mathfrak{p} R_{\mathfrak{p}}} M_{\mathfrak{p}}$. Thus $\operatorname{depth}_{\mathfrak{p}} M = \operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$. ☑

Corollary. For $\mathfrak{p} \in \operatorname{Supp} M$, $\operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$ is the smallest integer r such that $\mu_r(\mathfrak{p}, M) \neq 0$.

PROOF: It was shown above that $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_R(R/\mathfrak{p}, M)_{\mathfrak{p}}$. and also that $\operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$ is the smallest i such that $\operatorname{Ext}^i_R(R/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0$. Thus $\operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$ is the smallest i such that $\mu_i(\mathfrak{p}, M) \neq 0$. ☑

In an earlier corollary it was shown that if $M_{\mathfrak{p}}$ has finite injective dimension, then inj. $\dim M_{\mathfrak{p}}$ is the largest r such that $\mu_r(\mathfrak{p}, M) \neq 0$. Another corollary stated that if M has finite injective dimension, then inj. $\dim M_{\mathfrak{p}} = \operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}}$. Combining these observations with the corollary above and the standard fact that $\operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}} \leq \operatorname{height}_M \mathfrak{p}$ and the fact that $\operatorname{height}_M \mathfrak{p} \leq$ inj. $\dim M_{\mathfrak{p}}$ (because of the Hauptlemma), we get the following result.

Corollary. If $\mathfrak{p}$ is a prime in $\operatorname{Supp} M$ then $\operatorname{depth}_{\mathfrak{p}} M_{\mathfrak{p}} \leq \operatorname{height}_M \mathfrak{p} \leq$ inj. $\dim M_{\mathfrak{p}}$, with equality if $M_{\mathfrak{p}}$ has finite injective dimension.

Definition. An R-module M is a **Gorenstein module** if $\operatorname{depth}_{\mathfrak{p}} M = $ inj. $\dim M_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ in $\operatorname{Supp} M$. Equivalently (see the preceding corollary), M is Gorenstein if and only if M has finite injective dimension. If R is a Gorenstein module, then we call R a **Gorenstein ring**.

Lemma. For M to be a Gorenstein module, it suffices that $\operatorname{depth}_{\mathfrak{p}} M = $ inj. $\dim M_{\mathfrak{p}}$ for all *maximal* ideals $\mathfrak{p}$ in $\operatorname{Supp} M$.

Theorem. If R is a ring with finite Krull dimension (for instance if R is local [and noetherian]), then R is a Gorenstein ring if and only if R has finite injective dimension. Furthermore, in this case the injective dimension of R is the same as its Krull dimension.

PROOF: If R is a Gorenstein ring, then inj. $\dim R_{\mathfrak{m}} = \operatorname{depth}_{\mathfrak{m}} R = \operatorname{height} \mathfrak{m} < \infty$ for all maximal ideals $\mathfrak{m}$. Since the injective dimension of R is the supremum of inj. $\dim R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$, it follows that if R has finite Krull dimension then it also has finite injective dimension.

Conversely, if R has finite injective dimension, then so does $R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$. By a result above, inj. dim $R_{\mathfrak{p}} = \text{depth}_{\mathfrak{p}} R = \text{height } \mathfrak{p}$. Thus R is a Gorenstein ring and since inj. dim R is the maximum of inj. dim $R_{\mathfrak{p}}$ over all prime ideals $\mathfrak{p}$, thus inj. dim $R = \text{Krull dim } R$. ☑

Corollary. If $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals with $\mathfrak{q} \subseteq \mathfrak{p}$ and if $R_{\mathfrak{p}}$ is a Gorenstein ring, then so is $R_{\mathfrak{q}}$.

PROOF: Since injective resolutions localize, if $R_{\mathfrak{p}}$ has finite injective dimension than so does $R_{\mathfrak{q}}$. ☑

Note. A not necessarily commutative noetherian ring Λ has finite injective dimension if and only all Λ-modules with finite projective dimension also have finite injective dimension.

PROOF: If Λ-modules with finite projective dimension also have finite injective dimension, then in particular Λ, which is projective, has finite injective dimension. On the other hand, suppose that Λ has finite injective dimension. Then all projective Λ-modules have finite injective dimension, and it follows by induction that all Λ-modules with finite projective dimension have finite injective dimension. ☑

Theorem. Let $\mathfrak{p}$ be a prime in Supp M. The following conditions are equivalent:

(1) $\text{depth}_{\mathfrak{p}R_{\mathfrak{p}}} M_{\mathfrak{p}} = \text{inj. dim } M_{\mathfrak{p}}$.

(2) $\mu_i(\mathfrak{p}, M) \neq 0$ only when $i = \text{height}_M \mathfrak{p}$.

(3) $M_{\mathfrak{p}}$ is a Gorenstein module.

PROOF: As seen previously, $\text{depth}_{\mathfrak{p}} M$ is the smallest integer r such that $\mu_r(\mathfrak{p}, M) \neq 0$ and inj. dim M is the largest r such that $\mu_r(\mathfrak{p}, M) \neq 0$ and $\text{depth}_{\mathfrak{p}} M_{\mathfrak{p}} \leq \text{height}_M \mathfrak{p} \leq \text{inj. dim } M_{\mathfrak{p}}$. This shows that (1) and (2) are equivalent. Furthermore, the lemma above asserts that $M_{\mathfrak{p}}$ is a Gorenstein module if and only if $\text{depth}_{\mathfrak{p}} M_{\mathfrak{p}} = \text{inj. dim } M_{\mathfrak{p}}$, i. e. (1) and (3) are equivalent. ☑

Definition. An R-module M is **Cohen-Macauley** if $\text{depth}_{\mathfrak{p}} M = \text{height}_M \mathfrak{p}$ for all primes $\mathfrak{p}$ in Supp M.

Proposition. Gorenstein modules are Cohen-Macauley.

PROOF: See the previous theorem. ☑

Proposition. If M is Cohen-Macauley then Ass M consists of the minimal primes in Supp M.

PROOF: A prime ideal $\mathfrak{p} \in \text{Supp } M$ belongs to Ass M if and only if it consists of zero divisors on M, i. e. if and only if $\text{depth}_{\mathfrak{p}} M = 0$. If M is Cohen-Macauley, then these are precisely the prime ideals with height 0, i. e. the minimal primes in Supp M. ☑

Proposition. M is Cohen-Macauley if and only if for all $\mathfrak{p} \in \text{Supp } M$, $\mu_i(\mathfrak{p}, M) = 0$ for $i < \text{height}_M \mathfrak{p}$.

PROOF: As seen previously, $\text{depth}_{\mathfrak{p}} M$ is the smallest i such that $\mu_i(\mathfrak{p}, M) \neq 0$. ☑

Corollary. For M to be Cohen-Macauley, it suffices that $\operatorname{depth}_{\mathfrak{m}} M = \operatorname{height}_M \mathfrak{m}$ for all *maximal* ideals $\mathfrak{m} \in \operatorname{Supp} M$.

Corollary. If M is Cohen-Macauley then $\operatorname{Supp} M$ is catenary.

Lemma. If $R_{\mathfrak{p}}$ does not have finite injective dimension then $\mu_i(\mathfrak{p}, M) \neq 0$ for all $i \geq \operatorname{height} \mathfrak{p}$.

PROOF: If there exists a prime $\mathfrak{q}$ properly contained in $\mathfrak{p}$ such that inj. dim $R_{\mathfrak{q}} = \infty$, then this follows by induction from the Hauptlemma. On other hand, if inj. dim $R_{\mathfrak{q}} < \infty$ for all $\mathfrak{q} < \mathfrak{p}$, then inj. dim $R_{\mathfrak{q}} = \operatorname{height} \mathfrak{q}$. So if $i \geq \operatorname{height} \mathfrak{p}$ and E_i is the i$^{\text{th}}$ term in the injective resolution of R, then $(E_i)_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \subsetneqq \mathfrak{p}$, i.e. $\mu_i(\mathfrak{q}) = 0$. But $(E_i)_{\mathfrak{p}} \neq 0$, so we conclude that $\mu_i(\mathfrak{p}) \neq 0$. $\boxed{\checkmark}$

Lemma. Suppose that $x_1 \ldots, x_n$ is both a regular R-sequence and a regular M-sequence. Let I be the idea generated by $x_1, \ldots, x_n$. Let E be a minimal injective resolution of M and let $E_i' = \operatorname{Hom}_R(R/I, E_{i+n})$. Then E' is a minimal injective resolution for the R/I-module M/IM.

PROOF: E_i' is injective since if X is an R/I-module, then there is a natural isomorphism

$$\operatorname{Hom}_{R/I}(X, E_i') = \operatorname{Hom}_{R/I}(X, \operatorname{Hom}_R(R/I, E_{i+n})) \approx \operatorname{Hom}_R(X, E_{i+n}),$$

showing that $\operatorname{Hom}_{R/I}(\_, E_i')$ is an exact functor.

By induction, it suffices to consider the case where $n = 1$ and x_1 is regular on both R and M. Then proj. dim $R/(x) = 1$, so $\operatorname{Ext}_R^i(R/(x), M) = 0$ for $i > 1$. Also $\operatorname{Ext}_R^1(R/(x), M) = M/xM$. Thus

$$0 \to M/xM \to E_1' \to \ldots$$

is exact. $\boxed{\checkmark}$

Corollary. If I is the ideal generated by a regular R-sequence contained in $\mathfrak{p}$ which is also a regular M-sequence, then $\mu_i(\mathfrak{p}/I, M/IM) = \mu_{i+s}(\mathfrak{p}, M)$. In particular, if M is a Gorenstein [Cohen-Macauley] R-module, then M/IM is a Gorenstein [Cohen-Macauley] R/I-module. Conversely, if M/IM is a Gorenstein module then so is $M_{\mathfrak{p}}$.

Theorem. For any prime ideal $\mathfrak{p}$ of R, the following statements are equivalent:

(1) $R_{\mathfrak{p}}$ is a Gorenstein ring.

(2) $\operatorname{height} \mathfrak{p} = \operatorname{depth}_{\mathfrak{p} R_{\mathfrak{p}}} R_{\mathfrak{p}} = $ inj. dim $R_{\mathfrak{p}}$.

(3) $R_{\mathfrak{p}}$ has finite injective dimension.

(4) $\mu_i(\mathfrak{p}) = 0$ for all $i > \operatorname{height} \mathfrak{p}$.

(5) $\mu_i(\mathfrak{p}) = 0$ for at least one $i > \operatorname{height} \mathfrak{p}$.

(6) $\mu_i(\mathfrak{p}) = 0$ for all $i < \operatorname{height} \mathfrak{p}$ and $\mu_h(\mathfrak{p}) = 1$ for $h = \operatorname{height} \mathfrak{p}$.

(7) For all primes $\mathfrak{q} \subseteq \mathfrak{p}$, $\mu_i(\mathfrak{q}) = 0$ for $i \neq \operatorname{height} \mathfrak{q}$ and $\mu_i(\mathfrak{q}) = 1$ for $i = \operatorname{height} \mathfrak{q}$.

PROOF: We have seen the equivalence of (1), (2), and (3) above. Furthermore, a lemma above shows that if $R_{\mathfrak{p}}$ is not Gorenstein (i.e. inj. dim $R_{\mathfrak{p}} = \infty$), then $\mu_i(\mathfrak{p}) \neq 0$ for all $i > \operatorname{height} \mathfrak{p}$. On the other

hand, if $R_\mathfrak{p}$ is Gorenstein then inj. dim $R_\mathfrak{p} = $ height $\mathfrak{p}$, so $\mu_i(\mathfrak{p}) = 0$ for all $i > $ height $\mathfrak{p}$. Thus (1), (4) and (5) are equivalent.

$(1) \Rightarrow (6)$: Suppose that R is Gorenstein, and without loss of generality suppose that $\mathfrak{p}$ is maximal. Let $h = $ depth$_\mathfrak{p}$ $R = $ height $\mathfrak{p}$. Then $\mu_i(\mathfrak{p}) = 0$ for $i < h$. Let I be an ideal generated by a maximal regular R-sequence (necessarily of length h) in I. Then by the corollary above, R/I is a Gorenstein ring of dimension 0 and $\mu_h(\mathfrak{p}) = \mu_0(\mathfrak{p}/I, R/I)$. But this says that R/I is an injective R/I-module. By a corollary near the beginning of the chapter, this implies that $R/I = E((R/I)/(\mathfrak{p}/I)) = E(R/\mathfrak{p})$, so that $\mu_h(\mathfrak{p}) = \mu_0(\mathfrak{p}/I, R/I) = 1$.

$(6) \Rightarrow (1)$: By an earlier proposition, the statement that $\mu_i(\mathfrak{p}) = 0$ for $i < $ height $\mathfrak{p}$ is equivalent to the fact that $R_\mathfrak{p}$ is Cohen-Macauley. Assuming this to be true, the exists a regular R-sequence in $R_\mathfrak{p}$ with length equal to height $\mathfrak{p}$. Let I be the ideal in $R_\mathfrak{p}$ generated by this R-sequence. Then by the previous Corollary, $\mu_0(\mathfrak{p}R_\mathfrak{p}/I) = \mu_h(\mathfrak{p})$, where $h = $ height $\mathfrak{p}$. If $\mu_h(\mathfrak{p}) = 1$, this says that I is an irreducible ideal in $R_\mathfrak{p}$, and so by a result at the beginning of the Chapter, $R_\mathfrak{p}/I$ is self-injective. Thus in particular $R_\mathfrak{p}/I$ is a Gorenstein ring. It then follows from the previous corollary that $R_\mathfrak{p}$ is a Gorenstein ring.

$(6) \Leftrightarrow (7)$: This is now clear from the equivalent of (1), (4), and (6) plus the fact that a localization of a Gorenstein ring is a Gorenstein ring. ☑

Definition. A **system of parameters** for a local ring R is a sequence $x_1, \ldots, x_n$ contained in the maximal ideal $\mathfrak{m}$ such that $n = $ height $\mathfrak{m}$ and $\mathfrak{m}$ is the only prime ideal containing $x_1, \ldots, x_n$ (or, equivalently, Ass $R/(x_1, \ldots, x_n) = \{\mathfrak{m}\}$).

Proposition. Every noetherian local ring has a system of parameters.

Proposition. A noetherian local ring R is Cohen-Macauley if and only if every system of parameters is a regular R-sequence.

Lemma. Let R be a local ring of dimension n. Then $\mu_n(\mathfrak{m}) = 1$ if and only if some [every] system of parameters generates an irreducible ideal in R.

PROOF: Let I be generated by a system of parameters. Then Ass $R/I = \{\mathfrak{m}\}$ and so $E(R/I) \approx E(R/\mathfrak{m})^t$, where $t = \mu_0(R/I)$. Now $\mu_n(I) = 1$ if and only if $\mu_0(R/I) = 1$, i.e. if and only if $E(R/I)$ is indecomposable. As shown by a proposition in the beginning of the chapter, this is true if and only if I is irreducible. ☑

Theorem. Let R be a Cohen-Macauley ring with maximal ideal $\mathfrak{m}$. The following are equivalent.

(1) R is Gorenstein.

(2) R is Cohen-Macauley and some system of parameters generates an irreducible ideal.

(3) Every system of parameters in R generates an irreducible ideal.

(4) $\mu_i(\mathfrak{m}) = 0$ for $i < n$ and $\mu_1(\mathfrak{m}) = 1$.

(5) For every prime ideal $\mathfrak{q}$ in R, $\mu_i(\mathfrak{q}) = 0$ for $i < $ height $\mathfrak{q}$ and $\mu_h(\mathfrak{q}) = 1$ for $h = $ height $\mathfrak{q}$.

(6) If E_i is the ith term of a minimal injective resolution for R, then E_i is the direct sum of the injective envelopes of $R/\mathfrak{q}$ for all primes $\mathfrak{q}$ with height i.

(7) R is Cohen-Macauley and for all **finite length** modules M, there is a natural isomorphism

$$\operatorname{Ext}_R^n(M, R) \approx \operatorname{Hom}_R(M, E(k)).$$

PROOF: As shown earlier, R is Cohen-Macauley if and only if $\mu_i(\mathfrak{m}) = 0$ for $i < \operatorname{height} \mathfrak{m}$. Thus the equivalence of (1), (2), (3), and (4) follows from the preceding Lemma and the Theorem before it.

Lemma. Let M be a Cohen-Macauley module and I an ideal such that $IM \neq M$. Then $\operatorname{depth}_I M$ is the minimum of $\operatorname{height}_M \mathfrak{q}$ for all primes $\mathfrak{q}$ in $\operatorname{Supp} M$ which do not contain I.

Theorem. Let M be a finitely generated module over a local ring R. Then M is Gorenstein if and only if it has finite injective dimension and every regular R-sequence is also a regular M-sequence. In this case, M is also Cohen-Macauley.

Definition. If M is an R-module then **Cousin Complex** $C^*(M)$ is defined inductively as follows: $C^n = 0$ for $n < 0$. We write $M^0 = M$ and $C^0 = \coprod \{M_{\mathfrak{p}}^0 \mid \operatorname{height}_M \mathfrak{p} = 0\}$. Let $\beta^0 \colon M^0 \to C^0$ be the canonical map.

Now if M^{n-1}, C^{n-1} and β^{n-1} have been defined, let $M^n = \operatorname{Coker} \beta^{n-1}$, $C^n = \coprod \{M_{\mathfrak{p}}^n \mid \operatorname{height}_M \mathfrak{p} = n\}$ and let $\beta^n \colon M^n \to C^n$ be the canonical map. Let d^n be the composite of β^n with the canonical map from C^{n+1} to $M^{n+1} = \operatorname{Coker} \beta^{n+1}$:

$$C^n \longrightarrow M^{n+1} \xrightarrow{\ \beta^{n+1}\ } C^{n+1}.$$

Theorem. (1) $\{C^*(M), d^*\}$ is a cochain complex.

(2) If $M_{\mathfrak{p}}^n \neq 0$ then $\operatorname{height}_M \mathfrak{p} \geq n$.

(3) If $\operatorname{height}_M \mathfrak{p} = n - 1$ then every element of $(M^{n-1})_{\mathfrak{p}}$ is annihilated by some power of $\mathfrak{p}$.

(4) $H^n(C) \approx \operatorname{Ker} \beta^{n+1}$ and $\operatorname{height}_M \mathfrak{p} \geq n + 2$ for all $\mathfrak{p} \in \operatorname{Supp} H^n(C)$.

(5) If x is an element of R which annihilates M or annihilates $C^n(M)$, then x annihilates $C^t(M)$ for all $t \geq n$.

(6) If x is an element of R such that $xM = M$ or $xC^n(M) = C^n(M)$, then $xC^t(M) = C^t(M)$ for all $t \geq n$.

(7) If M is non-zero and finitely generated and if $x \in R$ is regular on M, then $xC^n = C^n$ for all n.

(8) If $n > \dim M$ then $C^n = 0$.

PROOF: (1) is clear.

(2) By induction. The map $C^{n-1} \to M^n$ is surjective so if $M_{\mathfrak{p}}^n \neq 0$ then $C_{\mathfrak{p}}^{n-1} \neq 0$. By the constuction of C^{n-1}, it follows that there exist primes $\mathfrak{q}$ with $\operatorname{height} \mathfrak{q} = n - 1$ such that $(M_{\mathfrak{q}}^{n-1})_{\mathfrak{p}} \neq 0$. Then $\operatorname{height} \mathfrak{p} \geq n - 1$ by induction. Now if $\operatorname{height} \mathfrak{p} = n - 1$ then

$\varnothing \neq \mathrm{Ass}(M_{\mathfrak{p}}^{n-1})_{\mathfrak{q}} \subseteq \{\mathfrak{q}\} \cup \{\mathfrak{p}\}$ so $\mathfrak{q} = \mathfrak{p}$, since the primes in $\mathrm{Ass}\, M^{n-1}$ have height at least $n-1$. Thus if $\mathrm{height}\,\mathfrak{p} = n-1$ then the only non-zero term $M_{\mathfrak{q}}^{n-1}$ in the coproduct defining C^{n-1} is $M_{\mathfrak{p}}^{n-1}$. This means that $\beta^{n-1}\colon M_{\mathfrak{p}}^{n-1} \to C_{\mathfrak{p}}^{n-1}$ is the identity map and so $M_{\mathfrak{p}}^{n} = 0$.

(3) If $\mathrm{height}_M \mathfrak{p} = n$ and $\mathfrak{p} \in \mathrm{Supp}\, M_{\mathfrak{p}}^{n-1}$, then $\mathfrak{p}$ must be minimal in $\mathrm{Supp}\, M_{\mathfrak{p}}^{n-1}$, so that $M_{\mathfrak{p}}^{n-1}$ is $\mathfrak{p}$-primary.

(4) $H^n(C) = (\mathrm{Ker}\, d^n)/d^{n-1}(C^{n-1}) = (\mathrm{Ker}\, d^n)/\beta^n(M^n) \approx \mathrm{Ker}\, \beta^{n+1} \subseteq M^{n+1}$. Now as shown in the proof of (2), if $\mathrm{height}\,\mathfrak{p} = n+1$ then localizing the map $\beta^{n+1}\colon M^{n+1} \to C^{n+1}$ at $\mathfrak{p}$ yields the identity map, so that $H^n(C)_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ with $\mathrm{height}\,\mathfrak{p} \le n+1$.

(5) Easy.

(6) Clear.

(7) If x is regular on M and $\mathrm{height}_M \mathfrak{p} = 0$ then $\mathfrak{p} \in \mathrm{Ass}\, M$, so $x \notin \mathfrak{p}$ and thus $xM_{\mathfrak{p}} = M_{\mathfrak{p}}$. Thus $xC^0 = C^0$. Now apply (6).

(8) Trivial. $\boxed{\checkmark}$

Theorem. For any multiplicative set S, the identity map on $S^{-1}M$ induces an isomorphism of complexes from $S^{-1}(C(M))$ to $C(S^{-1}M)$.

PROOF: Construct the isomorphism recursively. $\boxed{\checkmark}$

Remark. If $\mathfrak{p}$ has height n in $\mathrm{Supp}\, M$ then $C_{\mathfrak{p}}^i(M) = 0$ for $i > n$ and $C_{\mathfrak{p}}^n(M) \approx M_{\mathfrak{p}}$.

Lemma. Let B be a finitely generated modules such that $\mathrm{height}_M \mathfrak{p} \neq n$ for all $\mathfrak{p} \in \mathrm{Supp}\, B$. Then $\mathrm{Ext}_R^i(B, C^n(M)) = 0$ for all i.

PROOF: Since $C^n(M) = \coprod\{M_{\mathfrak{p}}^n \mid \mathrm{height}_M \mathfrak{p} = n\}$ and B is finitely generated, it follows that $\mathrm{Ext}_R^i(B, C^n(M)) = \coprod\{\mathrm{Ext}_R^i(B, M_{\mathfrak{p}}^n) \mid \mathrm{height}_m \mathfrak{p} = n\} = \coprod\{Ext^i(B_{\mathfrak{p}}, M_{\mathfrak{p}}^n) \mid \mathrm{height}_M \mathfrak{p} = n\} = 0$. $\boxed{\checkmark}$

Theorem. Suppose that β^0 induces an isomorphism $M \approx H^0(C(M))$, and that $H^i(C(M)) = 0$ for all $i \le n-2$. Let B be a finitely generated R-module such that $\mathrm{height}_m \mathfrak{p} \ge n$ for all $\mathfrak{p} \in \mathrm{Supp}\, B$. Then

$\mathrm{Ext}_R^i(B, M) = 0$ for $i \le n-1$,

$Extr^n(B, M) \approx \mathrm{Hom}_R(B, M^n)$,

$\mathrm{Ext}_R^j(B, M) \approx \mathrm{Ext}^{j-n}(B, M^n)$ for $j \ge n$.

PROOF: By item (4) of the previous theorem, the sequence $0 \to M^i \to C^i \to M^{i+1} \to 0$ is exact for $i \le n-1$. Furthermore $\mathrm{Ext}_R^i(B, C^r) = 0$ for all i whenever $r \le n-1$. $\boxed{\checkmark}$

Theorem. The following statements are equivalent:

(1) $C(M)$ is acyclic and β^0 induces an isomorphism from M to $H^0(C)$.

(2) $\text{Ext}_R^i(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \text{Supp}\, M$ with $\text{height}_M \mathfrak{p} > i$.

(3) M is Cohen-Macauley.

(4) If B is a finitely generated module such that $\text{height}_M \mathfrak{p} \geq n$ for all $p \in \text{Supp}\, B$, then $\text{Ext}_R^i(B, M) = 0$ for all $i < n$.

PROOF: It is easy to see that (1) implies (4), and (4) implies (2), and (2) is equivalent to (3).

(2) $\Rightarrow$ (1): We will show by complete induction that the maps β^i are all monic. Suppose now that $\beta^1, \ldots, \beta^{n-1}$ are monic and suppose $x \neq 0 \in \text{Ker}\, \beta^n = H^{n-1}(C)$. Choose $\mathfrak{p} \in \text{Ass}\, R_x$. Now if $H^{n+1}(C)_\mathfrak{p} \neq 0$ then $\text{height}_m \mathfrak{p} \geq n+1$, so that $\text{Ext}_R^n(R/\mathfrak{p}, M) \approx \text{Hom}_R(R/\mathfrak{p}, M^n) \neq 0$, a contradition. Thus $\text{Ker}\, \beta^n = 0$. $\boxed{\checkmark}$

Lemma. For all n, C^n is an essential extension of $\beta^n(M^n)$.

Theorem. Let M be a finitely generated R-module. The following assertions are equivalent:

(1) $C^*(M)$ is an injective resolution for M.

(2) For all finitely generated R-modules B and $0 \leq i < j$, $\text{Ext}_R^i(\text{Ext}_R^j(B, M), M) = 0$.

(3) M is a Gorenstein module.

(4) For all primes $\mathfrak{p} \in \text{Supp}\, M$, $C(M)_\mathfrak{p}$ is an injective resolution for the $R_\mathfrak{p}$-module $M_\mathfrak{p}$.

(5) M is Cohen-Macauley and if $n = \dim M$ then $C^n(M)$ is injective.