A COURSE IN HOMOLOGICAL ALGEBRA
Chapter **: Tor, Flatness, and Purity

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Proposition. If Λ is a K-algebra and M a Λ-module, then the functor $M \otimes_\Lambda \_$ is left adjoint to $\mathrm{Hom}_K(M, \_)$.

Corollary. Tensor products preserve colimits.

Definition. Flat modules.

Lemma. Projective modules are flat.

PROOF: Free modules are flat.

Definition. Tor.

Theorem. Tor commutes with (filtered) direct limits and coproducts.

Theorem. [Bourbaki, Commutative Algebra I, Lazard, Bull. Soc. Math. France **97**(1969,81–128]. The following are equivalent:

(1) M is flat.

(2) $X \to Y$ monic implies $X \otimes_\Lambda M \to Y \otimes_\Lambda M$ monic.

(3) For every ideal I, the multiplication map $I \otimes_\Lambda M \to M$ is monic.

(4) $\mathrm{Tor}_1(\Lambda/I, M) = 0$ for all finitely generated right ideals I.

(5) Whenever $a_j \in M$ and $\ell_{ij} \in \Lambda$ are such that $\sum_j \ell ija_j = 0$ for all i, then there exist $y_k \in M$ and $r_{ij} \in \Lambda$ such that

$$m_j = \sum_k r_{ij} y_k$$

$$\sum_j \ell_{ij} r_{jk} = 0 \qquad \forall\, i,\ k.$$

(6) For every map φ from a finitely generated Λ-module B into M, there exists homomorphism from a finitely generated free Λ-module F into M such that the following diagram commutes:

$$\begin{array}{ccc} B & \longrightarrow & F \\ & & \downarrow \\ & & M \end{array} \qquad \text{(Insert diagonal arrow for } \varphi.\text{)}$$

(7) M is a direct limit of finitely generated free modules.

Corollary. Finitely presented flat modules are projective.

Theorem. Let M be a flat Λ-module.

(1) Every regular element of Λ is regular on M.

(2) If I and J are right ideals, then $IM \cap JM = (I \cap J)M$.

Theorem. If $R \to A$ is a flat morphism of commutative rings and M is a finitely generated R-module, then $\operatorname{ann}(S \otimes_R M) = (\operatorname{ann}_R M)S$.

Proposition. If Λ is a flat K-algebra then

$$\Lambda \otimes_K \operatorname{Tor}_n^K(\_\_, \_\_) \approx \operatorname{Tor}_n^\Lambda(\Lambda \otimes_K \_\_, \Lambda \otimes_K \_\_).$$

Theorem. If R is a commutative ring, then M is flat if and only if $M_\mathfrak{p}$ is flat for all maximal ideals $\mathfrak{p}$.

Theorem. Let Λ be a quasi-local ring with maximal ideal J. Let M be a finitely presented left Λ-module. Then the following are equivalent:

(1) M is free.

(2) $\operatorname{Tor}_1^\Lambda(\Lambda/J, M) = 0$.

(2) The multiplication map $J \otimes_\Lambda M \to JM$ is monic.

Semi-Hereditary Rings and Prüfer Domains

Definition. A ring Λ is **left semi-hereditary** if every finitely generated left ideal is projective.

Theorem. Λ is left semi-hereditary if and only if all finitely generated submodules of projective modules are projective.

For noetherian rings the concept of "hereditary" and "semi-hereditary" coincide.

Theorem. If Λ is left semi-hereditary then $\operatorname{Tor}_n^\Lambda = 0$ for $n \geq 2$.

Definition. A **Prüfer domain** is a semi-hereditary integral domain.

A Prüfer domain is the non-noetherian (or not necessarily noetherian) analog of a dedekind domain.

Theorem. An integral domain is a Prüfer domain if and only if all finitely generated torsion free modules are projective.

Theorem. A module over a Prüfer domain is flat if and only if it is torsion free.

Theorem. If M and N are modules over a Prüfer domain R, then

$$\mathrm{Tor}_1^R(M,N) \approx \mathrm{Tor}_1^R(\mathbf{t}M,N),$$

where $\mathbf{t}M$ is the torsion submodule of M. Furthermore, $\mathrm{Tor}_1^R(M,N)$ is a torsion module.

Von Neumann Regular Rings

Definition. A ring is **von Neumann regular** if for every element a there exists an element x in the ring such that $axa = a$. In other words,

$$(\forall a \in \Lambda) \quad a \in a\Lambda a.$$

This condition definitely seems to be on the technical side. However it's actually much more natural than it appears, as the following theorem shows. Recall that a left ideal of Λ is generated by an idempotent if and only if it is a summand of Λ (considered as a left Λ-module).

Theorem. The following conditions for a ring Λ are equivalent:

(1) Λ is von Neumann regular.

(2) Every principal left idea is generated by an idempotent.

(3) Every finitely generated left ideal is generated by an idempotent.

(4) Every right Λ-module is flat.

(5) Every cyclic right Λ-module is flat.

PROOF: (1) $\Rightarrow$ (2): If $axa = a$ then xa is an idempotent and the principal left ideal Λa is generated by xa.

(2) $\Rightarrow$ (3): Consider a left ideal I generated by $a_1, \ldots, a_n$. By induction on the number of generators, we may suppose that there exists an idempotent f such that $\Lambda a_1 + \cdots + \Lambda a_{n-1} = \Lambda f$. Then by the modular law we get

$$\Lambda a_1 + \cdots + \Lambda a_n = \Lambda f \oplus (I \cap \Lambda(1-f))$$

and the rest is easy.

(3) $\Rightarrow$ (4): If suffices to see that $\mathrm{Tor}_1^\Lambda(M, \Lambda/I) = 0$ for all finitely generated left ideals I. But (3) implies that such an I is a summand of the ring, so Λ/I is projective.

(4) $\Rightarrow$ (5): Clear.

(5) $\Rightarrow$ (1): Let a be an element of Λ. Then by assumption $\Lambda/\Lambda a$ is flat. This yields an exact sequence

$$\mathrm{Tor}_1^\Lambda(a\Lambda,\, \Lambda/a\Lambda) = 0 \longrightarrow a\Lambda \otimes_\Lambda \Lambda a \longrightarrow a\Lambda \xrightarrow{\ \beta\ } a\Lambda \otimes_\Lambda \Lambda/\Lambda a \to 0\,.$$

Since $\mathrm{Ker}\,\beta = a\Lambda a$, we see that $a\Lambda \otimes_\Lambda \Lambda a \approx a\Lambda a$. The inclusion $a\Lambda \hookrightarrow \Lambda$ induces an exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & a\Lambda a & \longrightarrow & a\Lambda & \longrightarrow & a\Lambda \otimes_\Lambda (\Lambda/\Lambda a) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \gamma\big\downarrow & & \\ 0 & \longrightarrow & \Lambda a & \longrightarrow & \Lambda & \longrightarrow & \Lambda/\Lambda a & \longrightarrow & 0\,. \end{array}$$

Since $\Lambda/\Lambda a$ is flat, γ is monic and the left-hand square must be a pull back, in other words $a\Lambda a = \Lambda a \cap a\Lambda$. Thus $a \in a\Lambda a$, as required. ☑

Theorem. A commutative ring is von Neumann regular if and only if it has Krull dimension one and has no non-trivial nilpotent elements.

PROOF: ($\Rightarrow$): Easy.

($\Leftarrow$): It suffices to prove that $R_\mathfrak{p}$ is von Neumann regular for every prime ideal $\mathfrak{p}$. But since $R_\mathfrak{p}$ is zero-dimensional, $\mathfrak{p}R_\mathfrak{p} = \mathrm{nilrad}(R_\mathfrak{p}) = 0$, so $R_\mathfrak{p}$ is a field. ☑

Faithfully Flat Modules

Definitions. A functor $T\colon \mathcal{C} \to \mathcal{D}$ is **faithful** if it preserves the distinctness of morphisms, i. e. $T(\varphi) \neq T(\psi)$ whenever φ and ψ are distinct morphisms from an object X to an object Y in $\mathcal{C}$.

Generator. Cogenerator.

Proposition. An exact functor T from one abelian category to another is faithful if and only if TA is non-zero whenever A is non-zero.

PROOF: ($\Rightarrow$): Clear.

($\Leftarrow$): Let $\alpha\colon X \to Y$ be a non-zero morphism and let A be the image, so that α factors as $X \twoheadrightarrow A \rightarrowtail Y$. Since T is exact, $T\alpha$ factors as $TX \twoheadrightarrow TA \rightarrowtail TY$. Since $A \neq 0$, by hypothesis $TA \neq 0$. Thus $T\alpha$ is not zero. ☑

Proposition. An additive functor between abelian categories is faithful if and only if it **reflects exactness**, i. e. if $X \to Y \to Z$ is not exact in the source category, then $TX \to TY \to TZ$ will not be exact.

PROOF: ($\Leftarrow$): Let $0 \neq \beta\colon B \to C$ be a non-zero morphism in the source category. Then the sequence $B \xrightarrow{1_B} B \xrightarrow{\beta} C$ is not exact. Thus

$$TB \xrightarrow{1_{TB}} TB \xrightarrow{T\beta} C$$

is not exact, so $T\beta \neq 0$.

($\Rightarrow$): Let T be faithful and consider a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in the source category. Now if $\beta\alpha \neq 0$, then $T(\beta\alpha) \neq 0$ since T is faithful, so clearly $TA \xrightarrow{T\alpha} TB \xrightarrow{T\beta} TC$ is not exact.

Suppose therefore that $\beta\alpha = 0$ but that $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is not exact. Thus (thinking in terms of modules), $\alpha(A) \subsetneq \operatorname{Ker}\beta$.

In more categorial terms, let $K \xrightarrow{\sigma} B$ be the kernel of β and $B \xrightarrow{\tau} L$ be the cokernel of α. Intuitively, $L = B/\alpha(A)$ and the fact that $\alpha(A) \subsetneq \operatorname{Ker}\beta$ can be expressed in categorical terms by saying that $\tau\sigma \neq 0$. Then since T is faithful, $T(\tau\sigma) \neq 0$. Now $\beta\sigma = 0$ so $T(\beta)T(\sigma) = 0$ (since T is additive), which says that the image of $T(\sigma)$ is contained in $\operatorname{Ker} T\beta$. Also, $T(\tau\alpha) = 0$, so $\operatorname{Image}(T\alpha) \subseteq \operatorname{Ker}(T\tau)$. **** Thus to show that $TA \xrightarrow{T\alpha} TB \xrightarrow{T\beta} TC$ is not exact, it will certainly suffice to show that

$$\operatorname{Image}(T\alpha) \subsetneq \operatorname{Image}(T\sigma) \subseteq \operatorname{Ker} T\beta.$$

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Theorem. Let T be a faithful additive functor from Λ-modules to Λ'-modules.

(1) Suppose that T preserves direct limits. Let me be M a Λ-module such that TM is finitely generated. Then M is finitely generated.

(2) Suppose that T preserves all colimits and that $T\Lambda$ is a finitely presented Λ'-module. Then M is finitely presented.

Definition. A left Λ-module M is **faithfully flat** if $\_ \otimes_\Lambda M$ is a faithful and exact functor.

Theorem. The following conditions are equivalent for a left Λ-module M.

(1) M is faithfully flat.

(2) A sequence $0 \to A \to B \to C \to 0$ of right Λ-modules is exact if and only if the induced sequence

$$0 \to A \otimes_\Lambda M \longrightarrow B \otimes_\Lambda M \longrightarrow C \otimes_\Lambda M \to 0$$

is exact.

(3) M is flat and for every non-trivial right Λ-module A, $A \otimes_\Lambda M \neq 0$.

(4) M is flat and for all [maximal] right ideals I, $IM \neq 0$.

Faithfully Flat Ring Extensions

Definitions. Faithfully flat algebra. Faithfully flat morphism of rings.

Theorem. Let $\gamma\colon K \to \Lambda$ be a K-algebra. The following conditions are equivalent.

(1) Λ is a faithfully flat K-algebra.

(2) Λ is flat, γ is monic, and $\Lambda/\gamma(K)$ is a flat K-module.

(3) Λ is flat and for every K-module G, the standard map $G \to \Lambda \otimes_K G$ is monic.

(4) Λ is flat and every ideal $\mathfrak{a}$ of K is the contraction of its extension to Λ, i.e. $\mathfrak{a} = \gamma^{-1}(\mathfrak{a}\Lambda)$ (or, more informally, $\mathfrak{a} = K \cap \mathfrak{a}\Lambda$).

(5) Λ is flat and every maximal ideal $\mathfrak{m}$ in K is the contraction of an ideal in Λ, i.e. there exists an ideal I in Λ such that $\mathfrak{m} = \gamma^{-1}(I)$.

Lemma. Let Λ be a flat K-algebra. Then the standard map $\Lambda_K \operatorname{Hom}_K(A,B) \to \operatorname{Hom}_\Lambda(\Lambda \otimes_K A,\, \Lambda \otimes_K B)$ is a monomorphism when A is finitely generated and an isomorphicm when A if finitely presented.

Theorem. Let Λ be a faithfully flat K-algebra and let G be a K-module.

(1) G is flat if and only if $\Lambda \otimes_K G$ is flat.

(2) G is finitely generated [presented] if and only if $\Lambda \otimes_K G$ is finitely generated [presented].

(3) G is a finitely generated projective K-module if and only if $\Lambda \otimes_K G$ is a finitely generated projective Λ-module.

Proposition. Let Λ be a K-algebra. If there exists a faithfully flat Λ-module M which is flat as a K-module, then Λ is a flat K-algebra.

Pure Submodules

We were led to the idea of flat modules by looking for those modules M such that whenever A is a submodule of a right Λ-module B, the induced map $A \otimes_\Lambda M \to B \otimes_\Lambda M$ is monic.

Another natural line of investigation is to look for pairs $A \subseteq B$ such that for every possible left Λ-module M, the induced map $A \otimes_\Lambda M \to B \otimes_\Lambda M$ is monic. When this is the case, we say that A is a **pure submodule** of B.

Proposition. $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for the category of abelian groups.

Proposition. Let $T = \operatorname{Hom}_{\mathbb{Z}}(\_\,,\, \mathbb{Q}/\mathbb{Z})$ taking right Λ-modules to left Λ-modules. A right Λ-module A is flat if and only if $T(A)$ is an injective Λ-module.

Definition. A sequence $0 \to A \to B \to C \to 0$ of right Λ-modules is called **pure exact** if for every left Λ-module M the induced sequence

$$0 \to A \otimes_\Lambda M \longrightarrow B \otimes_\Lambda M \longrightarrow C \otimes_\Lambda M \to 0$$

is exact. (Since the tensor product is right exact, if the original sequence is exact, the only issue in doubt is that $A \otimes_\Lambda M \to B \otimes_\Lambda M$ be monic.)

A is called a **pure submodule** of B if the natural sequence

$$0 \to A \xrightarrow{\subseteq} B \longrightarrow B/A \to 0$$

is a pure exact sequence.

Clearly every split exact sequence is pure exact. Also, since the tensor product preserves direct limits, it is not hard to see that a direct limit of pure exact sequences is pure exact.

The following theorem says that all pure exact sequences can be obtained as direct limits of split exact sequences.

Theorem. Let $0 \xrightarrow{\alpha} A \xrightarrow{\beta} B \to C \to 0$ be a sequence of right Λ-modules. The following are equivalent.

(1) The sequence is pure exact.

(2) The sequence is exact and the morphism $T(A) \otimes_\Lambda \alpha$ is monic.

(3) The sequence of Λ-modules $0 \to TC \to TB \to TA \to 0$ is split exact.

(4) The sequence $0 \to TC \to TB \to TA \to 0$ is pure exact.

(5) For every finitely presented right Λ-module F, the induced map $\mathrm{Hom}_\Lambda(F, B) \longrightarrow \mathrm{Hom}_\Lambda(F, C)$ is surjective.

(6) The sequence $0 \to A \to B \to C \to 0$ can be obtained as a direct limit of split exact sequences.

PROOF: $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 6) \Rightarrow 1)$.

To prove $4) \Rightarrow 5)$ consider the map $\sigma\colon TA \otimes_\Lambda F \to T(\mathrm{Hom}_\Lambda(F, A))$ given by $\sigma(\theta \otimes f)(\varphi) = \theta(\varphi(f))$. This map σ is an isomorphism if F is finitely presented. ☑

Corollary [P. M. Cohn]. Let N be a submodule of the left Λ-module M. Then N is a pure submodule of M if and only if the following condition is satisfied.

> Whenever there exist elements $n_1, \ldots, n_r \in N$, $m_1, \ldots, m_s \in M$, and $\ell_{ij} \in \Lambda$ such that for all i, $\sum_j \ell_{ij} m_j = n_i$, then there exist $m'_1, \ldots, m'_s \in N$ satisfying the same equations, i. e.
>
> $$(\forall i) \quad \sum_j \ell_{ij} m'_j = n_i.$$

Corollary. If N is a pure submodule of M and I if a right ideal in Λ, then $IM = M \cap IN$.

Corollary. A pure subring of a noetherian [artinian] ring is noetherian [artinian].

Theorem. If K is a pure subring of the center of Λ, then $\Lambda \otimes_K \_\_$ is a faithful functor from K-modules to Λ-modules.

Lemma. If K is a pure subring of the center of Λ and $\mathfrak{a}$ is an ideal in K, then $\mathfrak{a}$ is a pure K-submodule of $\mathfrak{a}\Lambda$.

PROOF: Suppose that there exist $a_i \in \mathfrak{a}$, $b_j \in \mathfrak{a}\Lambda$ and $k_{ij} \in K$ such that for all i,

$$\sum_j b_j k_{ij} = a_i,$$

Since $b_j \in \mathfrak{a}\Lambda$, we can write $b_j = \sum_m c_m \ell_{jm}$, for $c_m \in \mathfrak{a}$. Then

$$\sum \sum_{j,m} c_m \ell_{jm} k_{ij} = a_i.$$

Since K is pure in Λ, there exist $\ell'_{jm} \in K$ such that $\sum \sum_{j,m} c_m \ell'_{jm} k_{ij} = a_i$. Now let

$$b'_j = \sum_m c_m \ell'_{jm} \in \mathfrak{a}\,.$$

Theorem. Let K be a pure subring of the center of Λ and let G be a K-module.

(1) G is flat if and only if $\Lambda \otimes_K G$ is a flat Λ-module.

(2) G is finitely generated [finitely presented] if and only if $\Lambda \otimes_K G$ is a finitely generated [finitely presented] Λ-module.

(3) G is a finitely generated projective module if and only if $\Lambda \otimes_K G$ is a finitely generated projective Λ-module.

PROOF: (1)($\Leftarrow$): Suppose that $\Lambda \otimes_K G$ is a flat Λ-module. Let $\mathfrak{a}$ be an ideal in K and consider the diagram

$$\begin{array}{ccccc} & & \mathfrak{a} \otimes_K G & \longrightarrow & G \\ & & \big\downarrow{\scriptstyle \text{monic}} & & {\scriptstyle \text{monic}}\big\downarrow \\ \mathfrak{a}\Lambda \otimes_K G & \xrightarrow{\approx} & \mathfrak{a}\Lambda \otimes_\Lambda \Lambda \otimes_K G & \xrightarrow{\text{monic}} & \Lambda \otimes_K M\,. \end{array}$$

(Insert diagonal arrow.)

Theorem. Let N be a submodule of a **flat** Λ-module M. The following are equivalent.

(1) N is a pure submodule of M.

(2) $IN = N \cap IM$ for all finitely generated right ideals I of Λ.

(3) M/N is flat.

Corollary. A pure submodule of a flat module is flat.

Theorem. If Λ is left noetherian, then pure submodules of injective left Λ-modules are injective.

Theorem. A push-out or pull-back of a pure exact sequence is pure exact, as is the Baer sum of two pure exact sequences. Thus there is a subfunctor $\operatorname{Pext}^1_\Lambda(\_\,,\_\,)$ of $\operatorname{Ext}^1_\Lambda(\_\,,\_\,)$ corresponding to the pure exact sequences.

Theorem. If R is a commutative ring, then a sequence $0 \to A \to B \to C \to 0$ of R-modules is pure exact if and only if for every maximal ideal $\mathfrak{p}$, the induced sequence

$$0 \longrightarrow A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0$$

is pure exact.