Chapter 10, continued.

Section 2. Simple Interest.

1. A loan of $10,000 collects simple interest each year for 5 years. At the end of that time, a total of $12,000 is paid back. Determine the APR for this loan.

Solution: The interest is $12,000 - $10,000 = $2000 over 5 years, so is $2,000/5 = $400 per year. The APR = $400/$10,000 = .04 = 4%.

2. You buy for $4000 a bond paying 3.5% simple interest. What is it worth after 10 years?

Solution: It is worth $4000 + 10 x .035 x $4000 = $4000 + 10 x $140 = $4000 + $1400 = $5400

Section 3. Compound Interest.

Suppose whenever we "compound" we receive interest not only on the principal, but also on all interest we have earned so far.

F(t) = future value at age t years.

F(0) = P = present value (principal)

**ANNUAL COMPOUNDING FORMULA**

The future value $F$ of $P$ dollars compounded annually for $t$ years at an APR of $R\%$ is given by

$$F = P(1 + r)^t$$

$F(1)/F(0) = 1 + r$ is called the common ratio.
Problem: Suppose $1000 is invested in a savings account with an APR of 3%, compounded annually. What is the future value of the account in 4 years?

Solution: $\ r = .03, \ so \ 1+r = 1.03 \ (\text{common ratio}).$
1 year: $F(1) = 1.03 x 1000 = 1030$
2 years: $F(2) = 1.03 F(1) = 1.03 x 1030 = 1060.90$
3 years: $F(3) = 1.03 F(2) = 1.03 x 1060.90 = 1092.727$
4 years: $F(4) = 1.03 F(3) = 1.03 x 1092.727 = 1125.5088$
Rounding off: $1125.51$

After 50 years? Keep doing what we did above 50 times, or use: $F(50) = 1000 x 1.03^{50} = 4383.91$
Here we use a calculator (or computer) with a power button to compute $1.03^{50}$, or $a^b$ for any numbers $a$ and $b$. You will find such a calculator very useful for the online homework, but you won't need it for the exam. (For some calculators you can compute $a^b$ using $\exp(b \ln(a)) = e^{b \ln(a)}$.)

Problem: A sum of money is invested in a CD paying an APR of 2.5%, compounded annually. At the end of 5 years you get a check for $2715.38. How much was the original principal?

Solution: $r = .025, \ so \ 1 + r = 1.025 \ (\text{common ratio}).$
$F(5) = 2715.38 = P x 1.025^5$
So $P = 2715.38 / 1.025^5$, so divide 2715.38 by 1.025 five times (or use a calculator which can compute $1.025^5$ directly).
Get $P = 2400$. 
More about compounding.

1000 is invested in a retirement account at 6% annual interest (interest is paid once a year, at the end of the year). How much money is in the account after 5 years?

If compound yearly:
F(N) = amount of money in account after N years
   = 1000 x 1.06^N  (1.06^N means 1.06^N)
F(5) = 1000 * (1.06)^5 = $1,338.26 (rounded off)

Suppose that the interest is compounded monthly. After one month, the interest paid would be
1000 * .06 / 12 = 1000 * 0.005 = $5.

The term periodic interest rate refers to the fraction .06/12 = .005 of interest paid each month, or to the corresponding percentage .5%.

After N months, the future value is F(N) = 1000 * (1.005)^N

After 5 years = 60 months, we will have

F(60) = 1000 * (1.005)^60 = $1,348.85

Notice that we end up with about $10 more money if the interest is compounded monthly. We would get even more if the interest were compounded daily. The periodic interest rate would be

.06/365=.00016438 (so the common ratio is 1.00016438),

the number of days (pretend there are no leap years) = 5 * 365 = 1825, so the amount of money after 5 years would be
F(1825) = 1000 * (1.00016438)^1825 = $1,349.82 (about $1 more).
Section 4. Geometric Sequences.

Start with a number P. Multiply by a number c repeatedly:

The numbers P, c P, c^2 P, c^3 P, \ldots, c^N P are called a geometric sequence. Here are a couple.

P = 3, c = 2: 3, 6, 12, 24, 48, ...
P = 2, c = 3: 2, 6, 18, 54, 162, ...

c is the ratio of two successive numbers in the sequence, it is called the \textbf{common ratio}.

If P is the amount invested with periodic interest rate r, then the future values P, F(1), F(2), F(3), ... are a geometric sequence with common ratio c = 1 + r

The Geometric Sum Formula

Suppose we have a geometric sequence F(n) = c^n P
We want a formula for adding the future values.
Say we want to compute X = P + c P + c^2 P + ... + c^(100)P.
Then c X = c( P + c P + c^2 P + ... + c^(100)P)
\hspace{1cm} = c P + c^2 P + ... + c^(100) P + c^(101) P
and (c-1) X = cX - X = c^(101) P - P = (c^(101) - 1) P,
so X = P (c^(101) - 1)/(c - 1).

More generally:

THE GEOMETRIC SUM FORMULA

\[ P + cP + c^2P + \ldots + c^{N-1}P = P\left(\frac{c^N - 1}{c - 1}\right) \]

5 + 2 \times 5 + \ldots 2^{10} \times 5 = 5 \left(\frac{2^{11} - 1}{2-1}\right) = 5(2048 - 1) = 10235