Chapter 10, continued.

Section 3. Compound interest

**General Compounding Formula**

The future value of $P$ dollars in $t$ years at an APR of $R\%$ compounded $n$ times a year is

$$F = P \left(1 + \frac{r}{n}\right)^{nt}$$

**General Compounding Formula (Version 2)**

The future value $F$ of $P$ dollars compounded a total of $T$ times at a periodic interest rate $p$ is

$$F = P(1 + p)^T$$

Section 4. Geometric sequences

**The Geometric Sum Formula**

$$P + cP + c^2P + \cdots + c^{N-1}P = P \left(\frac{c^N - 1}{c - 1}\right)$$
Section 5. Deferred Annuities

**Fixed deferred annuity:** equal payments made or received over regular time intervals, so as to produce a lump-sum payment in the future.

Book's problems:
63) Starting at age 25, Markus invests $2000 at the beginning of each year in an IRA with an APR of 7.5% compounded annually. How much money will there be in Markus's retirement account when he retires at the age of 65? (Assume that his 40th and last deposit generates interest for one year).

Solution: principal $P = 2000$, periodic (annual) interest rate $p = .075$ number of years $= 40$

1st payment (age 25) has future value at age 65 of $2000 \times 1.075^{40}$
2nd payment (age 26) has future value at age 65 of $2000 \times 1.075^{39}$
3rd payment (age 27) has future value at age 65 of $2000 \times 1.075^{38}$ etc.
40th payment (age 64) has future value at age 65 of $2000 \times 1.075$

The sum of the future value at age 65 of all these payments is

$$2000 \times 1.075 + 2000 \times 1.075^2 + \ldots + 2000 \times 1.075^{40} =$$

$$2000 \times 1.075(1 + 1.075 + \ldots + 1.075^{39}) =$$

$$2000 \times 1.075 \left( \frac{1.075^{40} - 1}{1.075 - 1} \right) \quad \text{by the geometric sum formula}$$

$$= \frac{2150 \times (1.075^{40} - 1)}{.075} = \$488,601.52$$
Here the $2150 is the future value of the last payment (the $2000 he invested at age 64 is worth $2150 when he retires).

The periodic (in this case annual) interest rate is $p = .075$ and the common ratio is $1+p = .075$. The number of time periods is $T=40$.

The general formula is:

---

**FIXED DEFERRED ANNUITY FORMULA**

The future value $F$ of a fixed deferred annuity consisting of $T$ payments of $SP$ having a periodic interest of $p$ (written in decimal form) is

$$F = L \left[ \frac{(1+p)^T - 1}{p} \right]$$

where $L$ denotes the future value of the last payment.

---

65) Starting at age 25, Markus invests $2000 at the end of each year in an IRA with an APR of 7.5% compounded annually. How much money will there be in Markus's retirement account when he retires at the age of 65? (Assume that his 40th and last deposit generates no interest).

The only difference from the previous example is that the payments are made at the end rather than the beginning of the year.

Future values at age 65 of the payments are:
- 1st payment (end of age 25) $2000 \times 1.075^{39}$
- 2nd payment (end of age 26) $2000 \times 1.075^{38}$, etc.
- 40th payment (end of age 64) $2000$

Total received at age 65 is
- $2000 + 2000 \times 1.075 + \ldots + 2000 \times 1.075^{39} =$
- $2000 \left( 1 + 1.075 + \ldots + 1.075^{39} \right) =$
- $2000 \left(1.075^{40} - 1\right)/.075 = \$454,513.04$
If we directly apply the Fixed Deferred Annuity Formula, the only difference between the two examples is that the future value of the last payment is $2000 instead of $2000 \times 1.075 = $2150.

69) Celine deposits $400 at the end of each month into an account that returns 4.5% annual interest (compounded monthly). At the end of three years she wants to take the money in the account and use it for a 20% down payment on a new home. What is the maximum price of a home that Celine will be able to buy?

\[ P = 400, \quad r = .045, \quad p = .045/12 = 0.00375, \quad T = 3 \times 12 = 36. \]

At the end of three years she will have:

\[ 400 \times 1.00375^{35} + 400 \times 1.00375^{34} + \ldots + 400 \times 1.0035 + 400 = 400 \times (1.00375^{36} - 1)/.00375 = $15,386.44 \]

This is supposed to be a 20% down payment on her house, so she can afford a house costing

\[ 15,386.44/.2 = 15,386.44 \times 5 = $76,932.20 \]
Section 6. Installment loans

An installment loan (also known as a fixed immediate annuity) is a series of equal payments made at equal time intervals for the purpose of paying off a lump sum of money received up front.

As in a deferred annuity, you make periodic equal payments at a fixed interest rate. The difference is that, instead of the future value of these payments at some time further in the future, we are interested in the present value of these payments; the present values add up to the amount we are borrowing now.

73a) Find the present value of an installment loan consisting of 25 annual payments of $3000 assuming an APR of 6% compounded annually. (Assume that the payments are made at the end of each year).

Solution. p = .06. The present values of the payments are:
1st: $3000/1.06, 2nd: $3000/1.06^2, \ldots 25th: $3000/1.06^{25}$

So the total present value of all the payments is

$$\frac{3000}{1.06} + \frac{3000}{1.06^2} + \ldots + \frac{3000}{1.06^{25}} =$$

$$\frac{3000}{1.06} \left( 1 + \frac{1}{1.06} + \ldots + \frac{1}{1.06^{24}} \right) =$$

$$\frac{3000}{1.06} \left( \frac{1}{1.06^{25}} -1\right)/ \left( \frac{1}{1.06} -1\right) = $38,350.07$$

This is similar to the annuity formula, with $1+p$ replaced by $q = 1/(1+p)$, and the "future value of the last payment" replaced by the "present value of the first payment".
The general formula is:

\[
P = Fq \left[ \frac{q^T - 1}{q - 1} \right]
\]

where \( q = \frac{1}{1 + p} \).

74a) Find the present value of a 60 month installment loan consisting of 60 monthly payments of $16 assuming an APR of 6% compounded monthly. (Assume that the payments are made at the end of each month.)

periodic interest rate \( p = \frac{.06}{12} = .005 \)

\( q = \frac{1}{1.005} = 0.9950248756 \)

Present value =

\[
\frac{16}{1.005} + \frac{16}{1.005^2} + \ldots + \frac{16}{1.005^{60}} =
\]

\[
\frac{16}{1.005} \left( 1 + \frac{1}{1.005} + \ldots + \frac{1}{1.005^{59}} \right) =
\]

\[
\frac{16}{1.005} \left( \frac{1}{1.005^{60}} - 1 \right) / \left( \frac{1}{1.005} - 1 \right) =
\]

$827.61