Sections 5. Probability spaces with equally likely outcomes

Suppose we have a random experiment with sample space \( S = \{o_1, \ldots, o_N\} \). Suppose the outcomes are equally likely.

Then \( \Pr(o_1) = \ldots = \Pr(o_N) = 1/N \).

Suppose an event \( E \) consists of \( K \) of these outcomes. Then \( \Pr(E) = K/N \).

p.581 # 52. Consider the random experiment of drawing 1 card out of a deck of 52 cards. Find the probability of each of the following events.

a) \( E_1 \): the card drawn is a queen
b) \( E_2 \): the card drawn is a heart
c) \( E_3 \): the card drawn is a face card

Answers: a) there are 4 queens, so \( \Pr(E_1) = 4/52 = .077 \)
b) there are 13 hearts, so \( \Pr(E_2) = 13/52 = .25 \)
c) there are \( 3 \times 4 = 12 \) face cards, so \( \Pr(E_3) = 12/52 = .23 \)
Consider the random experiment of drawing 2 cards out of a deck of 52 cards (the order of the cards doesn’t matter). Find the probability of each of the following events.

a) $E_1$: draw a pair of queens
b) $E_2$: draw a pair (a pair is two cards of the same value)

Answers: The sample space consists of all pairs of cards—the order doesn’t matter. The number of pairs of cards chosen from 52 cards is

$$52\binom{2} = \frac{52 \times 51}{2 \times 1} = 26 \times 51 = 1326.$$

a) The number of pairs of queens is $\binom{4}{2} = \frac{4 \times 3}{2 \times 1} = 6$, so

$$\Pr(E_1) = \frac{6}{1326} = .0045.$$

b) There are 13 values, so

$$\Pr(E_2) = \frac{13 \times 6}{1326} = .059$$

Suppose a student takes a 10-question true-false quiz and the student randomly guesses the answer for each question (so the probability the student gets the right answer equals the probability the student gets the wrong answer). Find the probability that the student gets

a) 10 right.
b) 0 right.
c) 9 right.
d) 9 or more right.

Answers: a) $1/(2^{10}) = 1/1024 = .001$ (by multiplication rule there are $2^{10}$ possible ways of answering the quiz, all equally likely; there is only one way to get 10 right)
b) $1/1024 = .001$ (only one way to get 10 wrong)
c) $10/1024 = .01$ (there are 10 choices for which is guessed wrong)
d) $(10 + 1)/1024 = .011$
p.582 # 56. Suppose that the probability of giving birth to a boy and the probability of giving birth to a girl are both .5. In a family of 4 children, what is the probability that
a) all 4 children are girls.
b) there are 2 girls and 2 boys.
c) the youngest child is a girl.

Answers: Typical outcomes are for example BBGG or GBGG. The number of possible outcomes is \(2^4 = 16\).
a) \(1/16 = .0625\)

b) How many ways can we have 2 girls and 2 boys? There are 4 births—we have to choose 2 of them for the boys. So the number of ways of doing this is \(\binom{4}{2} = 6\). So the probability is \(6/16 = .375\).

c) There are \(2^3 = 8\) possible genders for the three oldest children. So the probability that the youngest child is a girl is \(8/16 = \frac{1}{2} = .5\). This is the same as the probability of a single child being a girl. So the probability that this child is a girl is not affected by the gender of the other children.

Consider the random experiment of tossing a fair coin five times. Find the probability of each of the following events.
a) \(E_1\) : We toss no tails.
b) \(E_2\) : We toss at least one tail.
c) \(E_3\) : We toss twice as many heads as tails.

Answers: The sample space has size \(2^5 = 32\).

a) the only outcome is HHHHH, so \(\Pr(E_1) = 1/32 = .03125\)

b) every outcome is either in \(E_1\) or in \(E_2\) so
\(\Pr(E_2) = 1 - .03125 = .96875\)

c) can’t happen, the probability is 0.
Here are some poker questions. Recall that a standard deck of cards has four suits (Hearts, Diamonds, Spades and Clubs) and 13 values (Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2), making 4 * 13 = 52 cards.

A poker hand consists of five cards. The number of ordered poker hands is \( \binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5!} = 311,875,200 \).

The number of unordered poker hands is \( \frac{\binom{52}{5}}{5!} = \frac{311,875,200}{120} = 2,598,960 \).

A poker hand is called “four-of-a-kind” if it has all four of some value, and one other card. What’s the probability of being dealt a four-of-a-kind?

Answer: The four-of-a-kind could be of 2’s, 3’s, etc. Once you decide which of the 13 values, those four cards are determined. You then have to pick one card from the remaining 48 cards: 13 * 48 = 624. The probability of getting a four-of-a-kind is 624/2,598,960 = .00024 (about 240 per million—very small).

A poker hand is called “a flush” if it has all five cards of the same suit. How many flushes are there?

Answer: First choose the suit (there’s four of them) and then choose five from those thirteen:
\( 4 \times \binom{13}{5} = 4 \times 13 \times 12 \times 11 \times 10 \times 9 / 120 = 5148 \). The probability of getting a flush is 5148/2,598,960 = .002 (about 10 times better than a four-or-a-kind).
A poker hand is called “a full house” if it has three cards of one value and two cards of some other value. How many full houses are there?

Answer: First choose the value of the three cards, choose which three cards of that value, and then choose the value of the pair, and which two cards of that value:

\[13 \times \binom{4}{3} \times 12 \times \binom{4}{2} = 13 \times 4 \times 12 \times 6 = 3744.\] The probability of getting a full house is \[3744/2,598,960 = .00144\] (about 25% harder than getting a flush, but much easier than a four-or-a-kind).