Review:

**Election Methods**

**Plurality method:** the candidate with a plurality of votes wins.

**Plurality-with-elimination method (Instant runoff):** Eliminate the candidate with the fewest first place votes. Keep doing this until some candidate has a majority.

**Borda count method:** Assign points for the position each candidate finishes on each ballot; 0 points for last place, 1 for second-to-last place, 2 for third-to-last, etc. Whoever receives the most of these Borda points is the winner.

**Method of pairwise comparisons:** Compare each candidate to each other; whichever of the two candidates was more preferred by the voters gets one point. Add up the points for all of the comparisons. The candidate with the most points is the winner.

**Fairness Criteria**

**Majority Criterion:** If a candidate receives a majority of the first place votes, that candidate should win the election.

**Monotonicity Criterion:** If a candidate wins an election, and then we change some of the ballots, but only so as to increase the ratings on those ballots of the winning candidate, then that candidate should still win.

**Condorcet Criterion:** If a candidate wins all pairwise comparisons, that candidate should win the election.

**Criterion of Independence of Irrelevant Alternatives:** If a candidate wins, and then one of the losing candidates is eliminated, then the original winner still wins.

**Arrow’s Impossibility Theorem:** It is impossible to devise an election method, which satisfies all four fairness criteria.
None of these four election methods satisfy IIA. The only one to satisfy Condorcet is the method of pairwise comparisons, which satisfies three of the four criteria. Pairwise comparisons’ disadvantages include that it results in lots of ties, and if there are many candidates the number of pairwise comparisons can be very large. How large?

\[
1 + 2 + 3 + \ldots + n \\
+ n + n-1 + n-2 + \ldots + 1 \\
= n(n+1)
\]

So \(1 + 2 + 3 + \ldots + n = n(n+1)/2\).

Suppose there are \(N\) candidates. When making pairwise comparisons, the first candidate A has \(N-1\) comparisons, the second B has \(N-2\) not counting comparison to A, etc.

Then there are \(N-1 + N-2 + \ldots + 2 + 1 = (N-1)N/2\) pairwise comparisons.

<table>
<thead>
<tr>
<th>Number of candidates</th>
<th>Number of pairwise comparisons</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2*3/2=3</td>
</tr>
<tr>
<td>4</td>
<td>3*4/2=6</td>
</tr>
<tr>
<td>5</td>
<td>4*5/2=10</td>
</tr>
<tr>
<td>6</td>
<td>5*6/2=15</td>
</tr>
<tr>
<td>100</td>
<td>99<em>100/2=99</em>50=4950</td>
</tr>
</tbody>
</table>
1.6 Rankings

Often you want to know the ranking of all the candidates, rather than just who came in first. Each of the election methods we looked at produces a method for ranking. We will apply these methods to the example from the first lecture:

<table>
<thead>
<tr>
<th>Number of voters</th>
<th>3</th>
<th>2</th>
<th>4</th>
<th>Borda points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; choice</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>2</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; choice</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; choice</td>
<td>C</td>
<td>C</td>
<td>A</td>
<td>0</td>
</tr>
</tbody>
</table>

**Extended plurality method:** rank by first place (or choice) votes.

C is ranked 1<sup>st</sup>, A is 2<sup>nd</sup>, B is 3<sup>rd</sup>.

**Extended Borda count method:** Assign Borda points. Rank by number of Borda points.

A receives $3 \times 2 + 2 \times 1 + 1 \times 0 = 8$ points.
B receives $3 \times 1 + 2 \times 2 + 4 \times 1 = 11$ points.
C receives $3 \times 0 + 2 \times 0 + 4 \times 2 = 8$ points
Therefore B is 1<sup>st</sup>, and A and C are tied for 2<sup>nd</sup>.

**Extended method of pairwise comparisons:** Assign pairwise-comparison points. Rank by these points.

A versus B: 3 prefer A, 6 prefer B. B gets 1 point.
A versus C: 5 prefer A, 4 prefer C. A gets 1 point.
B versus C: 5 prefer B, 4 prefer C. B gets 1 point.
So A gets 1 point and B gets 2 points.
Therefore B is 1<sup>st</sup>, A is 2<sup>nd</sup>, and C is 3<sup>rd</sup>.
<table>
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<th>3</th>
<th>2</th>
<th>4</th>
</tr>
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<tbody>
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<td>1&lt;sup&gt;st&lt;/sup&gt; choice</td>
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<td>A</td>
<td>B</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; choice</td>
<td>C</td>
<td>C</td>
<td>A</td>
</tr>
</tbody>
</table>

**Extended plurality-with-elimination method:** Eliminate the candidate with the fewest first place votes. Keep doing this until some candidate has a majority. Rank by reverse order of elimination.

B has two first place votes, so is eliminated, leaving preference schedule:

<table>
<thead>
<tr>
<th>Number of voters</th>
<th>5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; choice</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; choice</td>
<td>C</td>
<td>A</td>
</tr>
</tbody>
</table>

So A is 1<sup>st</sup>, C is 2<sup>nd</sup>, B is 3<sup>rd</sup>.

There is another collection of methods for ranking called **recursive ranking methods**, which are discussed on pages 24-26. We will not study these and they will not be tested.
Weighted Voting Systems

Sometimes voters don’t have an equal vote. For example, in the Electoral College used to elect the President, there are 51 voters (the 50 states and DC), and each state gets a number of votes equal to its number of representatives and senators. In corporate shareholder’s elections, each shareholder gets as many votes as he or she has shares. The voters are called players, and the number of votes each player gets is called that player’s weight. We assume the players are voting yes or no on some motion. The number of yes votes needed to pass the motion is called the quota.

The total possible yes votes is the sum of the weights. The quota can’t be bigger than that, and must be more than half that (why?).
Here is an example showing how we describe a weighted voting system:

[17: 5, 5, 4, 2]
This means there are four players, whose weights are 5, 5, 4 and 2, and that the quota is 17. But the sum of the weights is 5+5+4+2=16, so it is impossible for a vote to be 17 or more, so it is impossible for a motion to pass. We don’t allow this.

[16: 5, 5, 4, 2]
In this case, a motion only passes if all four players vote for it. All four players have **veto power**.

[9: 5, 5, 4, 2]
In this case the motion passes if at least two of the first three players vote Yes. How the fourth player (called P4) votes is irrelevant, so P4 is called a **dummy**.

[8: 5, 5, 3, 3]
In this case, a motion could pass with equal votes for and against; we don’t allow this.

[8: 8, 3, 3, 1]
In this case, a motion only passes if P1 votes for it. P1 is called a **dictator**.

[9: 8, 3, 3, 1]
In this case, a motion only passes if P1 and any other player votes for it. P1 is not a dictator, but has veto power.