Section 5. Euler’s Theorems.

Recall: an Euler path or Euler circuit is a path or circuit that travels through every edge of a graph once and only once. The difference between a path and a circuit is that a circuit starts and ends at the same vertex, a path doesn't.

Suppose we have an Euler path or circuit which starts at a vertex $S$ and ends at a vertex $E$. Let $A$ be any other vertex. Every time we arrive at $A$ along one edge we must also leave $A$ along another edge; so there is a pair of edges at $A$ for each time we arrive at $A$:

\[
\begin{align*}
\text{A} & \quad \text{or} \quad \text{A} & \quad \text{or} \quad \text{A} & \quad \text{etc.}
\end{align*}
\]

Since we travel over all the edges, $A$ must have even degree (an even number of edges touching $A$: 2 or 4 or 6, etc.).

If it is an Euler circuit, then $S=E$. There will be one edge along which we start our journey, one along which we end it, and a pair of edges for every other time we pass through $S=E$. So $S=E$ also has even degree.

If it is an Euler path, then $S$ is different from $E$. There will be one edge at $S$ along which we start our journey, one at $E$ along which we end it, and a pair of edges for every other time we pass through $S$ or through $E$. So $S$ and $E$ will have odd degree.
We summarize:

**Euler’s Circuit Theorem.**

(a) If a graph has *any* vertices of odd degree, then it *cannot* have an Euler circuit.
(b) If a graph is *connected* and *every* vertex has even degree, then it has at least one Euler circuit. The Euler circuits can start at any vertex.

**Euler’s Path Theorem.**

(a) If a graph has *other than two* vertices of odd degree, then it *cannot* have an Euler path.
(b) If a graph is *connected* and has exactly *two* vertices of odd degree, then it has at least one Euler path. Every Euler path has to start at one of the vertices of odd degree and end at the other.

**Examples:**

![Graphs](image)

No Euler p. or c.   Euler p. no Euler c.   Euler c. no Euler p.

In each of these examples, count the number of edges, and add up all the degrees. What do you notice? Euler noticed this too:

**Euler’s Sum of Degrees Theorem.**

(a) The sum of the degrees of all the vertices of a graph equals twice the number of edges (and therefore must be an even number).
(b) The number of vertices of *odd* degree must be *even.*
Section 6. How to find an Euler path or circuit.

Fleury’s Algorithm:

1. First make sure the graph is connected, and the number of vertices of odd degree is either two or zero.
2. If none of the vertices have odd degree, start at any vertex. If two of the vertices have odd degree, start at one of these two.
3. Whenever you come to a vertex, choose any edge at that vertex that hasn’t been used yet to travel along next, except: do not travel through a bridge for the untraveled part, unless you have no alternative.
4. Label the edges in the order in which you travel them.
5. When you can’t travel any more, stop. (You are done!)

Example:

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C} \\
\text{E} \\
\text{D} \\
\text{F} \\
\end{array} \]

There are two vertices of odd degree, B and E. We can choose either one to start at. If we traveled: B, E, D, F, E, we would find ourselves stuck at E. The problem is that we traveled over edge BE, which was a bridge. We should have used either BA or BC. For example, B, A, C, B, E, D, F, E works!

Example:

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{D} \\
\text{E} \\
\text{C} \\
\text{F} \\
\end{array} \]

All the vertices have even degree, so we can start at any one. Choose A. Notice that there are no bridges. So let’s start traveling.
A, B, D, A ... whoops! We’re stuck. After we went: A, B, D, the untraveled part of the graph looked like:

Example:

```
A
  B
  C
  D
  E
F
```

The edge DA was a bridge of the untraveled part of the graph, so we can’t use it as long as there is an alternative, which there is: either DE or DF.

After ABD we can go to E or F; let's choose F:

Example:

```
A
  B
  C
  D
  E
F
```

The we must go to E. Note ED is now a bridge so we can't go to D, we can either go to B or C; let's choose C. Then we are forced to go to B, then E, then D, then A. So the following circuit works:

A, B, D, F, E, C, B, E, D, A.

Example:

```
A
  B
  C
  D
  E
F
```
Section 7. Eulerizing graphs.

Example (Königsberg bridges). There are four vertices of odd degree, so there cannot be an Euler path. How many bridges must we **recross** to make a circuit which crosses all bridges **at least** once?

![Graph of Königsberg bridges](image)

Draw just enough edges beside existing ones so that all vertices of odd degree change to having even degree. This is called Eulerizing the graph. Here we do it for the Königsberg bridges graph by adding an RD and an AL:

![Eulerized graph](image)

Now all vertices have even degree. We can start at any place, and make a circuit, but we have to cross bridge DR and the left bridge AL twice.