5.3 Higher derivatives, concavity and the second derivative test.

The second derivative of f is the derivative of f'.

It's written f''.

For example:

$f(x) = x^4 + 3x^2$. $f'(x) = 4x^3 + 6x$.

$f''(x) = 12x^2 + 6$. **You can continue:**

$f'''(x) = 24x$, $f^{(4)}(x) = 24$, $f^{(5)}(x) = 0$.

Leibniz notation: $y = \ln(x)$, $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$.

The second derivative is $\frac{d^2y}{dx^2} = -x^{-2} = \frac{-1}{x^2}$,

$\frac{d^3y}{dx^3} = 2x^{-3} = \frac{2}{x^3}$, **etc.**

We will use 2nd derivatives a lot, but rarely higher derivatives.

Motion. $s(t) =$ position at time t.

$v(t) = s'(t) =$ velocity.

$a(t) = v'(t) = s''(t) =$ acceleration.

So acceleration is the rate of change of velocity.

We mentioned Galileo's law:

if $s(t)$ is height above the ground,

$s_0 =$ initial height and $v_0 =$ initial velocity,

then $s(t) = -16t^2 + v_0 t + x_0$ ft.

So $v(t) = -32t + v_0$ ft/sec

and $a(t) = -32$ ft/sec/sec.

So Galileo's law implies constant acceleration.

By Newton, $F = ma$ so this is equivalent to constant force.

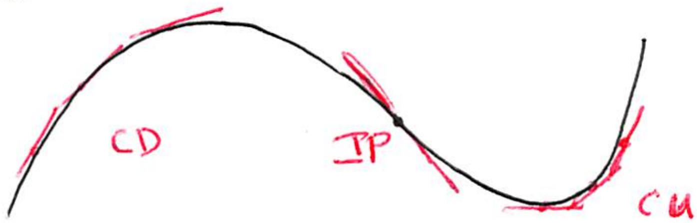
Why is the acceleration (and force) negative?

Geometric interpretation: concavity.

f is *concave up* (CU) on (a, b) if the graph lies above each tangent line at points $(x, f(x))$, $a < x < b$.

It is *concave down* (CD) if the graph lies below the tangent lines.

An *inflection point* (IP) is a point on the graph where the concavity changes.



Test for Concavity.

a) Suppose $f''(x) > 0$ on $a < x < b$. Then f is CU on (a,b).

b) Suppose $f''(x) < 0$ on $a < x < b$. Then f is CD on (a,b).

Example. $f(x) = x^2 - 2x$. $f'(x) = 2x - 2$.

$f''(x) = 2 > 0$, **so** f **is CU on** $(-\infty, \infty)$.

Example. $f(x) = x^3 - 3x^2$. $f'(x) = 3x^2 - 6x$.

$f''(x) = 6x - 6 = 6(x - 1) = 0$ **at** $x = 1$;

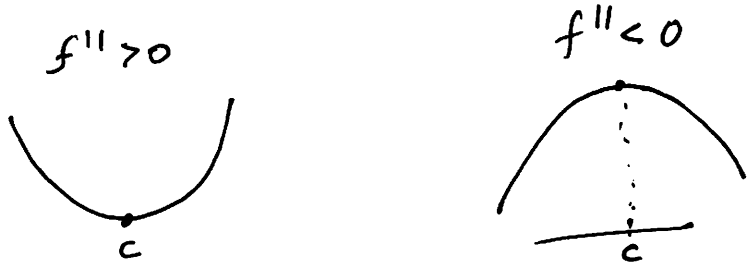
$f''(0) < 0$ **and** $f''(2) > 0$, **so**

f **is CD on** $(-\infty, 1)$ **and CU on** $(1, \infty)$; $(1, -2)$ **is an IP.**

Second Derivative Test. **Suppose c is a CN of f.**

If $f''(c) > 0$, then f has a relative minimum at c.

If $f''(c) < 0$, then f has a relative maximum at c.



Example: Find the relative extrema of $f(x) = 2x^3 - 9x^2$.

$f'(x) = 6x^2 - 18x = 6x(x-3)$. **The CN's are $x = 0, 3$.**

$f''(x) = 12x - 18$, $f''(0) = -18 < 0$ **so $(0,0)$ is a rel. max.,**

$f''(3) = 18 > 0$, **so $(3, -27)$ is a rel. min..**