Chapter 6

6.1 Absolute Extrema.

\( f \) has an absolute (or global) maximum at \( x = c \) if \( f(c) \geq f(x) \) for all \( x \) in the domain. It has an absolute minimum if \( f(c) \leq f(x) \) for all \( x \) in the domain.

\[ \text{Domain } [a, b] \]

Extreme Value Theorem. If \( f \) is continuous on a closed interval \([a, b]\), then \( f \) will have both an abs max and an abs min on \([a, b]\).
Finding absolute extrema on \([a, b]\):

1. Find all the CNs \(c\) in \((a, b)\).
2. Evaluate \(f(c)\) at these CNs.
3. Evaluate \(f(a)\) and \(f(b)\).
4. The largest of these values is the abs max. The smallest is the abs min.

**Example 1:** \(f(x) = x^2 - 2x\) on \([0, 3]\). \(f'(x) = 2x - 2 = 0\) at \(x = 1\).

\(f(1) = -1, f(0) = 0, f(3) = 3\), so \(f(1) = -1\) is the abs min, \(f(3) = 3\) is the abs max. Graph:

If the domain of \(f\) is not a closed interval \([a, b]\), then replace the endpoints by limits.

**Example 2:** \(f(x) = x + \frac{1}{x}\) on \((0, \infty)\). \(f'(x) = 1 - \frac{1}{x^2} = 0\) at \(x = 1\).

\(\lim_{x \to 0^+} f(x) = \lim_{x \to \infty} f(x) = \infty\). So \(f(1) = 2\) is the abs min, and there is no abs max.
We can find extrema using a graphing calculator or a computer. However, sometimes the critical numbers are hard to find on the calculator without computing derivatives. Also our function may involve parameters (constants not having definite values).

**Example 3:** The Ricker model for fish populations says that if $S$ is this year’s population, then next year’s population will be $f(S) = aS e^{-bS}$, where $a > 0$ and $b > 0$ are “parameters” (constants that depend on the fish species and on the environment). What population $S$ will result in the largest population next year?

$$f'(S) = aS e^{-bS}(-b) + ae^{-bS} = a(-bS + 1)e^{-bS} = 0 \text{ if } -bS + 1 = 0,$$

so $S = 1/b$. Note $f'(0) = a > 0$ and $f'(2/b) = -ae^{-2} > 0$, so $f$ has an abs max at $S = 1/b$.  

(1st derivative test)

**Example 4:** Suppose that a patient is given a dosage $x$ of some medication, and the probability of a cure is $P(x) = \frac{\sqrt{x}}{1 + x}$, $0 \leq x < \infty$. What dosage maximizes the probability of a cure?

$$P'(x) = \frac{(1 + x)^{3/2}x^{-1/2} - \sqrt{x}}{(1 + x)^2} \cdot 1 = \frac{1 + x - 2x}{2\sqrt{x}(1 + x)^2} = \frac{1 - x}{2\sqrt{x}(1 + x)^2} = 0$$

if $x = 1$.

Since $P'(x) > 0$ on $(0, 1)$ and $P'(x) < 0$ on $(1, \infty)$, the probability is maximized at $x = 1$.  

1st derivative test.
6.2 Applications of Extrema. "Max-Min Word Problems"

Identify the variables in the problem.

What limits must they satisfy?

What variable is to be maximized or minimized?

Express it as a function $f$ of one of the other variables.

Find the max or min as in the last section.

**Example 1:** A rectangular plot of ground is to be enclosed by a fence and then divided down the middle by another fence. Find the largest area that can be so enclosed by using 60 ft of fence total.

variables: $x$, $y$, $A$.

$0 \leq x, y$, $2x + 3y = 60$, so $y = 20 - \frac{2}{3}x$.

Maximize $A = xy = x(20 - \frac{2}{3}x) = 20x - \frac{2}{3}x^2$, domain $0 \leq x \leq 30$.

$A'(x) = 20 - \frac{4}{3}x = 0$ at $x = 15$.

$A = 0$ at the endpoints; $A = 150$ is the maximum when $x = 15$, $y = 10$. 
Example 2: Find the largest area of a rectangle with two sides on the positive $x$ and $y$ axes, and a vertex on the line $7x + 6y = 42$.

variables: $x, y$, maximize $A = xy$.

$0 \leq x, y$, $y = 7 - \frac{7}{6}x$, so $A = 7x - \frac{7}{6}x^2$.

$A' = 7 - \frac{7}{3}x = 0$ if $x = 3$ CN. Since $A'' = -\frac{7}{3} < 0$, $A$ has a max at $x = 3, y = \frac{7}{3}$.

(Or use endpoints $0 \leq x \leq 6$

$A(0) = 0$ and $A(6) = 0$ so max at $x = 3$.)