6.2 Applications of Extrema—continued.

Identify the variables in the problem.

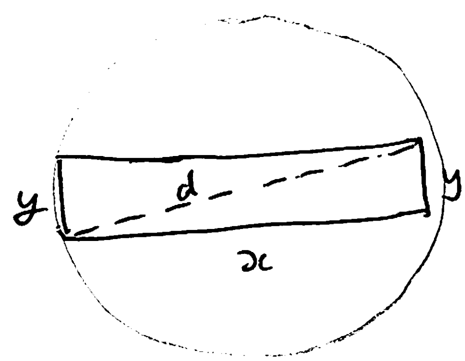
What limits must they satisfy?

What variable is to be maximized or minimized?

Express it as a function f of one of the other variables.

Find the max or min as in the last section 6.1.

Example 3. The strength S of a rectangular beam is proportional to its width x and the square of its depth y. The beam is cut from a log of diameter d. Choose x and y so that S is maximal.



variables: x, y, S.

$0 \le x, y$, $x^2 + y^2 = d^2$, **so** $y = (d^2 - x^2)^{1/2}$.

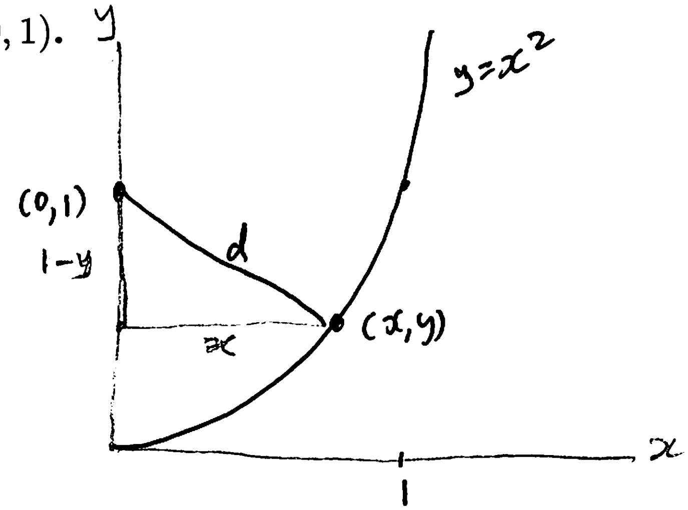
Maximize $S = kxy^2 = kx(d^2 - x^2) = k(d^2x - x^3)$, **domain** $0 \le x \le d$.

$S'(x) = k(d^2 - 3x^2) = 0$ **at** $x = d/\sqrt{3}$.

$S = 0$ **at the endpoints;**

S **is the maximum when** $x = d/\sqrt{3}$, $y = \sqrt{2/3}d$.

Example 4. Find the point (x, y) on the parabola $y = x^2$ (with $x \geq 0$) closest to the point $(0, 1)$.



variables: x, y, $d^2 = x^2 + (y-1)^2$, $0 \leq x, y$.

The minimum of d occurs at the same point as the minimum of $S = d^2$, and this is easier to work with.

So minimize $S = x^2 + (y-1)^2 = x^2 + (x^2-1)^2 = x^4 - x^2 + 1$,

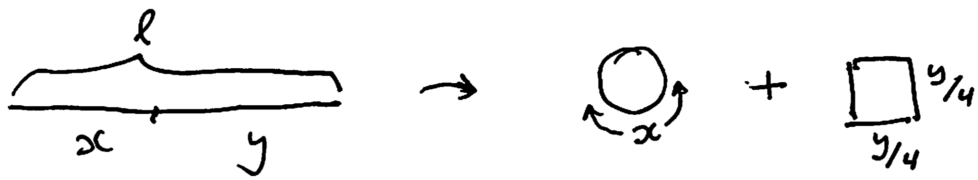
domain $0 \leq x$.

$S'(x) = 4x^3 - 2x = 2x(2x^2 - 1) = 0$ **at** $x = 0, 1/\sqrt{2}$.

$S' < 0$ **on** $(0, 1/\sqrt{2})$ **and** $S' > 0$ **on** $(1/\sqrt{2}, \infty)$; S **and hence** d **is the minimum at** $(1/\sqrt{2}, 1/2)$.

(We could instead compare $S(0)$, $S(1/\sqrt{2})$, $\lim_{x\to\infty} S(x)$**).**

Example 5. A string of length l is cut into 2 pieces; one piece is made into a circle and the other into a square. Where should the cut be made to maximize the area enclosed? to minimize?



variables: x, y, $A = \dfrac{x^2}{4\pi} + \dfrac{y^2}{16}$.

$0 \le x, y \le l$; $x + y = l$.

Maximize $A = \dfrac{x^2}{4\pi} + \dfrac{(l-x)^2}{16}$, domain $0 \le x \le l$.

$A'(x) = \dfrac{x}{2\pi} - \dfrac{(l-x)}{8} = \dfrac{x}{2\pi} + \dfrac{x}{8} - \dfrac{l}{8} = 0$ at $x = \dfrac{\pi l}{4+\pi}$.

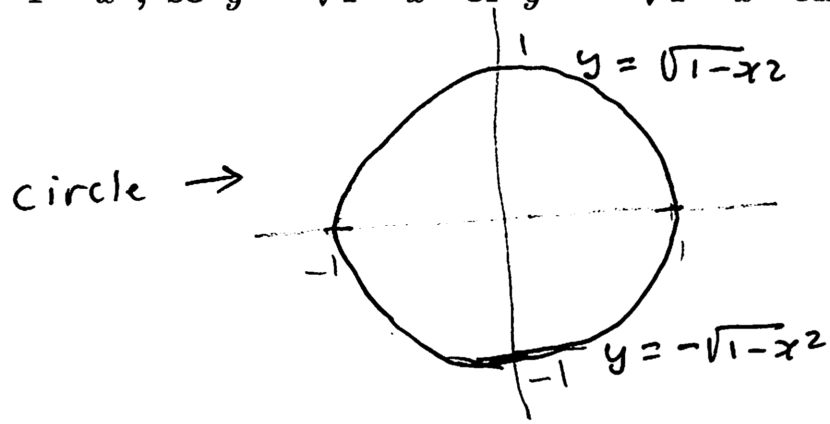
$A''(x) = \dfrac{1}{2\pi} + \dfrac{1}{8} > 0$ so the graph of A is CU, so

A has an absolute minimum at $x = \dfrac{\pi l}{4+\pi}$.

$A(0) = l^2/16 < A(l) = l^2/(4\pi)$, so the absolute maximum is when $x = l$ (use the whole string to make a circle).

6.3 Implicit Differentiation.

An equation like $x^2 + y^2 = 1$ (*) can be used to define y as a function of x: $y^2 = 1 - x^2$, so $y = \sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$ on $-1 \leq x \leq 1$.



We can then compute

$$\frac{dy}{dx} = \begin{cases} \frac{-x}{\sqrt{1-x^2}} & y > 0 \\ \frac{x}{\sqrt{1-x^2}} & y < 0 \end{cases}.$$

But we can also compute dy/dx directly from (*):

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}$, **which gives the above answer.**

$x^4 + xy + y^2 = 7$ is a curve. Find the slope at (1,2).

We could solve for y using the quadratic formula, but it's much easier to use implicit differentiation.

$$x^4 + xy + y^2 = 7$$

$$\frac{d}{dx}(x^4 + xy + y^2) = \frac{d}{dx}(7)$$

$$4x^3 + x\frac{dy}{dx} + 1 \cdot y + 2y\frac{dy}{dx} = 0$$

$$(4x^3 + y) + (x + 2y)\frac{dy}{dx} = 0$$

$$(x + 2y)\frac{dy}{dx} = -(4x^3 + y)$$

$$\frac{dy}{dx} = -\frac{4x^3 + y}{x + 2y} = -\frac{6}{5}$$ **at (1,2).**

Write the tangent line in slope-intercept form:

$$y - 2 = -\tfrac{6}{5}(x - 1)$$

$$y = -\tfrac{6}{5}x + \tfrac{16}{5}.$$