6.4 Related Rates.

Related rates problems: x, y functions of t

x is related to y by some formula

You know $\frac{dx}{dt}$; find $\frac{dy}{dt}$.

Example 1: $xy + \sin y = 1$, $\frac{dx}{dt} = 2$;

find $\frac{dy}{dt}$ when $y = \frac{\pi}{2}$ (so $x = \frac{2}{\pi}$).

$$x\frac{dy}{dt} + y\frac{dx}{dt} + \cos y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-y}{x + \cos y}\frac{dx}{dt} = \frac{-\frac{\pi}{2}}{\frac{2}{\pi}}2 = -\frac{\pi^2}{2}.$$

Example 2: Air is pumped into a spherical balloon at rate 2 cu.in./sec.

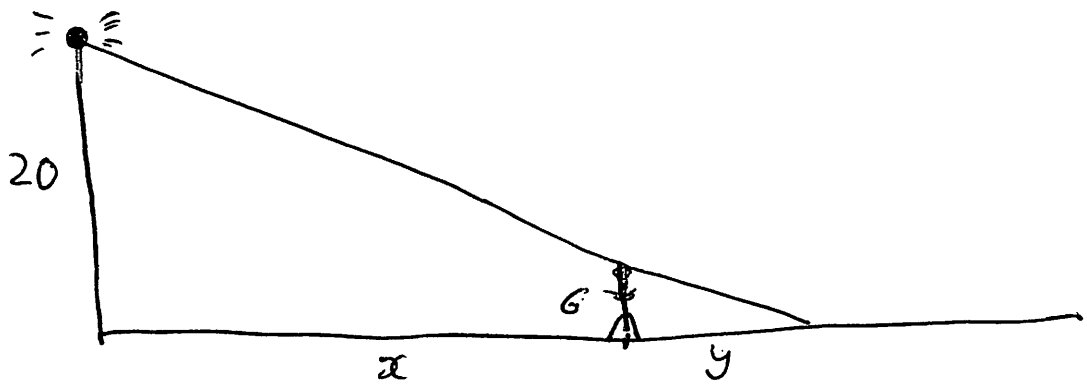
How fast is the radius increasing when $r = 1, 2, 3$?

$$V = \tfrac{4}{3}\pi r^3$$

$$2 = \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{2\pi r^2} = \frac{1}{2\pi}, \frac{1}{8\pi}, \frac{1}{18\pi}$$ **$r = 1, 2, 3$, respectively.**

Example 3: A man walks on a straight horizontal path away from a light that hangs 20 feet above the path. How fast does his shadow lengthen if he is 6 feet tall and walks at 140 ft/min?



Find $\frac{dy}{dt}$**. Given** $\frac{dx}{dt} = 140$**.**

By similar triangles:

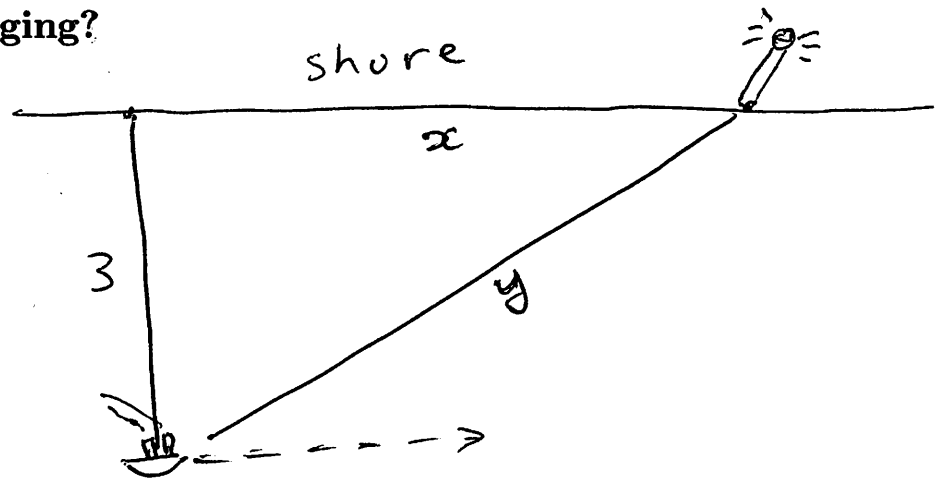
$$\frac{x+y}{20} = \frac{y}{6}$$

$$6x + 6y = 20y$$

$$y = \frac{3}{7}x$$

$$\frac{dy}{dt} = \frac{3}{7}\frac{dx}{dt} = 60 \text{ ft/min.}$$

Example 4: A ship is sailing east parallel to a straight shoreline at 20mph. The distance from the ship to the shore is 3 mi. There is a lighthouse on the shore NE of the ship. When the ship is 5 mi from the lighthouse, how fast is the distance to the lighthouse changing?



Given $\frac{dx}{dt} = -20$ **(negative because** x **is decreasing).**

Find $\frac{dy}{dt}$ **when** $y = 5$**.**

By the Pythagorean Theorem:

$x^2 + 3^2 = y^2$

When $y = 5$**,** $x = 4$**.**

$$2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt} = \frac{4}{5} \cdot (-20) = -16mph$$

Example 5:

In physics, the kinetic energy of a moving object is given by the formula $K = \frac{1}{2}mv^2$ where m is the mass and v is the velocity. Suppose an object with a mass of 2.00 kg is accelerating at a rate of $6.00\,m/s^2$. How quickly is the kinetic energy of the object increasing when the velocity is $23.0\,m/s$?

$$\frac{dK}{dt} = mv\frac{dv}{dt} = (2\,kg)(23\,m/s)(6\,m/s^2) = 276\,kg \cdot m^2/s^3.$$

7.1 Antiderivatives.

If $F'(x) = f(x)$, then $F(x)$ is called an antiderivative of $f(x)$. See table:

$F(x)$	1	x	x^2	x^3	x^n	$\sin x$	$\ln x$
$f(x) = F'(x)$	0	1	$2x$	$3x^2$	nx^{n-1}	$\cos x$	$1/x$

$x + 5$ is also an antiderivative of 1. So antiderivatives are not unique. If $F' = G' = f$, then $(F - G)' = 0$ so $F - G = C$ (on an interval). So antiderivatives differ by a constant.

The entire family of all antiderivatives of f is denoted $\int f(x)\,dx$, and equals $F(x)+C$, where F is any one antiderivative of f, and C varies over all real numbers (we say "C is an arbitrary constant").

If $n \neq -1$, $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$,

$\int x^{-1}\,dx = \ln x + C$,

$\int e^x\,dx = e^x + C$,

$\int \cos x\,dx = \sin x + C$, **etc.**

Rules: $\int f \pm g\,dx = \int f\,dx + \int g\,dx$ **and** $\int kf\,dx = k\int f\,dx$.

$\int 2x^4 - 5x + 1\,dx = \frac{2}{5}x^5 - \frac{5}{2}x^2 + x + C$,

$\int \dfrac{x^2+1}{\sqrt{x}}\,dx = \dfrac{x^3/3 + x}{\frac{2}{3}x^{3/2}} + C$, ?

$\int (x^3+1)^2\,dx = (x^4/4 + x)^3/3 + C$, ?

$\int e^{kx}\,dx = e^{kx}/k + C$