6.4 Related Rates.

Related rates problems: $x, y$ functions of $t$

$x$ is related to $y$ by some formula

You know $\frac{dx}{dt}$; find $\frac{dy}{dt}$.

Example 1: $xy + \sin y = 2, \frac{dx}{dt} = 2$;

find $\frac{dy}{dt}$ when $y = \frac{\pi}{2}$ (so $x = \frac{2}{\pi}$).

\begin{align*}
  x \frac{dy}{dt} + y \frac{dx}{dt} + \cos y \frac{dy}{dt} &= 0 \\
  \frac{dy}{dt} &= \frac{-y \frac{dx}{dt}}{x + \cos y} = \frac{-\frac{\pi}{2} \cdot 2}{\frac{2}{\pi}} = \frac{-\pi^2}{2}.
\end{align*}

Example 2: Air is pumped into a spherical balloon at rate 2 cu.in./sec.

How fast is the radius increasing when $r = 1, 2, 3$?

$V = \frac{4}{3} \pi r^3$

$2 = \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{2\pi r^2} = \frac{1}{2\pi} \cdot \frac{1}{8\pi} = \frac{1}{18\pi} r = 1, 2, 3$, respectively.
Example 3: A man walks on a straight horizontal path away from a light that hangs 20 feet above the path. How fast does his shadow lengthen if he is 6 feet tall and walks at 140 ft/min?

Find $\frac{dy}{dt}$. Given $\frac{dx}{dt} = 140$.

By similar triangles:

$$\frac{x + y}{20} = \frac{y}{6}$$

$$6x + 6y = 20y$$

$$y = \frac{3}{7}x$$

$$\frac{dy}{dt} = \frac{3}{7} \frac{dx}{dt} = 60 \text{ ft/min}.$$
Example 4: A ship is sailing east parallel to a straight shoreline at 20mph. The distance from the ship to the shore is 3 mi. There is a lighthouse on the shore NE of the ship. When the ship is 5 mi from the lighthouse, how fast is the distance to the lighthouse changing?

Given \( \frac{dx}{dt} = -20 \) (negative because \( x \) is decreasing).

Find \( \frac{dy}{dt} \) when \( y = 5 \).

By the Pythagorean Theorem:

\[ x^2 + 3^2 = y^2 \]

When \( y = 5 \), \( x = 4 \).

\[ 2x \frac{dx}{dt} + 0 = 2y \frac{dy}{dt} \]

\[ \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt} = \frac{4}{5} \cdot (-20) = -16 \text{ mph} \]
Example 5:

In physics, the kinetic energy of a moving object is given by the formula \( K = \frac{1}{2}mv^2 \) where \( m \) is the mass and \( v \) is the velocity. Suppose an object with a mass of 2.00 kg is accelerating at a rate of 6.00 \( m/s^2 \). How quickly is the kinetic energy of the object increasing when the velocity is 23.0 \( m/s \)?

\[
\frac{dK}{dt} = mv \frac{dv}{dt} = (2 \text{ kg})(23 \text{ m/s})(6 \text{ m/s}^2) = 276 \text{ kg} \cdot \text{m}^2/\text{s}^3.
\]

7.1 Antiderivatives.

If \( F'(x) = f(x) \), then \( F(x) \) is called an antiderivative of \( f(x) \). See the table:

\[
\begin{array}{c|cccccccc}
F(x) & 1 & x & x^2 & x^3 & x^n & \sin x & \ln x \\
\hline
f(x) = F'(x) & 0 & 1 & 2x & 3x^2 & nx^{n-1} & \cos x & \frac{1}{x} \\
\end{array}
\]

\( x + 5 \) is also an antiderivative of 1. So antiderivatives are not unique. If \( F' = G' = f \), then \( (F - G)' = 0 \) so \( F - G = C \) (on an interval). So antiderivatives differ by a constant.
The entire family of all antiderivatives of \( f \) is denoted \( \int f(x) \, dx \), and equals \( F(x) + C \), where \( F \) is any one antiderivative of \( f \), and \( C \) varies over all real numbers (we say "\( C \) is an arbitrary constant").

If \( n \neq -1 \), \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \),

\( \int x^{-1} \, dx = \ln x + C \),

\( \int e^x \, dx = e^x + C \),

\( \int \cos x \, dx = \sin x + C \), etc.

**Rules:** \( \int f \pm g \, dx = \int f \, dx + \int g \, dx \) and \( \int k \cdot f \, dx = k \int f \, dx \).

\( \int 2x^4 - 5x + 1 \, dx = \frac{2}{5}x^5 - \frac{5}{2}x^2 + x + C \),

\( \int \frac{x^2 + 1}{\sqrt{x}} \, dx = \frac{x^3/3 + x}{2\sqrt{x}^3/2} + C \), ?

\( \int (x^3 + 1)^2 \, dx = (x^4/4 + x)^3/3 + C \), ?

\( \int e^{kx} \, dx = e^{kx}/k + C \)