7.1 Antiderivatives—continued.

If $F'(x) = f(x)$**, then** $F(x)$ **is called an antiderivative of** $f(x)$**.** $\int f(x)\,dx = F(x) + C$**, where** F **is any one antiderivative of** f**, and** C **varies over all real numbers (we say "**C **is an arbitrary constant"). This is called and "indefinite integral".**

If $n \neq -1$**,** $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$**,**

$\int e^x\,dx = e^x + C$**,**

$\int \cos x\,dx = \sin x + C$**,**

$\int \sin x\,dx = -\cos x + C$**.**

$\int x^{-1}\,dx = \ln x + C$ — **this only holds for** $x > 0$**, more general is:**

$\int x^{-1}\,dx = \ln|x| + C$

Rules: $\int f \pm g\,dx = \int f\,dx \pm \int g\,dx$ **and** $\int kf\,dx = k\int f\,dx$**.**

$\int 2x^4 - 5x + 1\,dx = \frac{2}{5}x^5 - \frac{5}{2}x^2 + x + C$**,**

$\int \dfrac{x^2+1}{\sqrt{x}}\,dx = \dfrac{x^3/3 + x}{\frac{2}{3}x^{3/2}} + C$**, ?**

$\int (x^3+1)^2\,dx = (x^4/4+x)^3/3 + C$, ?

$\int e^{kx}\,dx = e^{kx}/k + C$

Motion problems again. $y(t)$ = **position at time** t, $v = x'$ = **velocity,** $a = v'$ = **acceleration.**

Newton's law: Force = mass times acceleration: $F = ma$

Assume force F **and hence** a **are constant:** $a = -g$

Then $v = -gt + C$ **and** $v_0 = v(0) = -g \cdot 0 + C = C$, **so**

$v = -gt + v_0$

Then $y = -gt^2/2 + v_0 t + C$, **and** $y_0 = y(0) = C$, **so**

$y = -gt^2/2 + v_0 t + y_0$. **Galileo's law.**

Suppose an object moves with acceleration $a(t) = 3 - 2t$, **and with starting position** $s(0) = 2$ **and starting velocity** $v(0) = 5$. **Find the position function** $s(t)$.

$v(t) = \int a(t)\,dt = \int 3 - 2t\,dt = 3t - t^2 + C = 3t - t^2 + 5$

$s(t) = \int v(t)\,dt = \int 3t - t^2 + 5\,dt = \frac{3}{2}t^2/2 - \frac{1}{3}t^3 + 5t + 2.$

7.2. Substitution

$\frac{d}{dx}((x^2+1)^{100}) = 100(x^2+1)^{99} \cdot 2x$ **chain rule**

Thus $\int 100(x^2+1)^{99} \cdot 2x\,dx = (x^2+1)^{100} + C$.

If we didn't already know the answer, we could find this antiderivative using a "u-substitution". In this case we would let

$u = x^2+1,\ du = 2x\,dx,$

$\int 100(x^2+1)^{99} \cdot 2x\,dx = \int 100u^{99}\,du$

$= u^{100} + C = (x^2+1)^{100} + C.$

In general, we look for a composition $g(f(x))$ **in our "integrand".**

$f(x)$ **is the "inside function" of the composition.**

We let $u = f(x)$ **and** $du = f'(x)\,dx$.

Using this we convert the original integral into a new one involving u **and** du **(we must entirely remove the original variable** x **and** dx**).**

Examples

$\int e^{5x}\,dx =$ **(let $u = 5x$ and $du = 5\,dx$, so $dx = \frac{1}{5}\,du$)**

$\int e^u \frac{1}{5}\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C.$

More generally, for any nonzero constant k, $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$.

$\int x^2 e^{x^3}\,dx =$ **(let $u = x^3$, $du = 3x^2dx$, so $dx = du/(3x^2)$)**

$$\int x^2 \frac{1}{3x^2} e^u\,du = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^{x^3} + C.$$

$$\int \frac{x^4}{x^5+1}\,dx =$$ **(let $u = x^5 + 1$, $du = 5x^4\,dx$)**

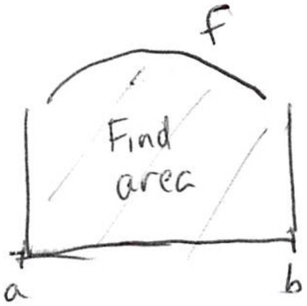
$$\int \frac{1}{u}\frac{1}{5}\,du = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|x^5+1| + C.$$

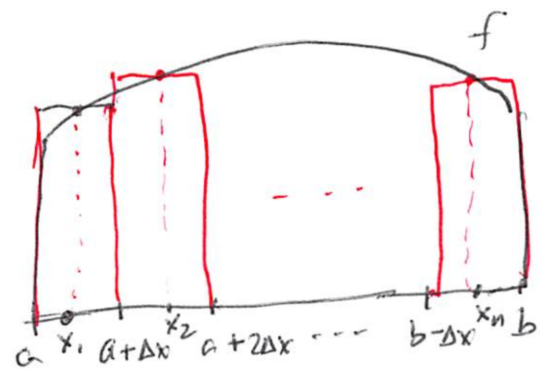
$$\int \frac{1}{x\ln^2 x}\,dx =$$

$$\int \frac{x}{(x+1)^3}\,dx =$$

$$\int x^3\sqrt{x^2+1}\,dx =$$

Suppose we want to compute the area bounded above by the graph $y = f(x)$, below by the x-axis, and on the sides by the vertical lines $x = a$ and $x = b$.





We can approximate this region by a polygonal region:

Choose an integer n, and we "partition" $[a, b]$ into n equal subintervals of length $\Delta x = (b - a)/n$;

the first subinterval is $[a, a + \Delta x]$, the second is $[a + \Delta x, a + 2\Delta x]$, etc., the nth subtinterval is $[b - \Delta x, b]$.

In each subinterval choose one point: x_1 in the first subinterval, x_2 in the second, etc., x_n in the last.

On each subinterval construct a rectangle whose base is the subinterval; for the ith subinterval the height of the rectangle is $f(x_i)$.

The area of this polygon is $\sum_{1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + \cdots + f(x_n)\Delta x$.

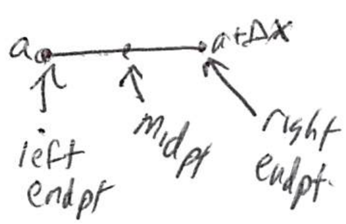
This sum is called a "Riemann sum".

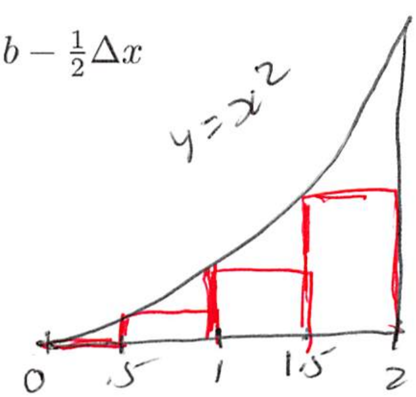
There are many ways of choosing the points x_i;

The "left-endpoint method" says to use left-endpoints of each subinterval: $x_1 = a$, $x_2 = a + \Delta x$, $\ldots$, $x_n = b - \Delta x$

The "right-endpoint method" says to use right-endpoints of each subinterval: $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\ldots$, $x_n = b$

The "midpoint method" says to use midpoints of each subinterval: $x_1 = a + \frac{1}{2}\Delta x$, $x_2 = a + \frac{3}{2}\Delta x$, $\ldots$, $x_n = b - \frac{1}{2}\Delta x$





Approximate the area bounded between $y = f(x) = x^2$, **the** x-**axis and** $x = 2$ **using** $n = 4$ **subintervals and the left endpoint method:**

$\Delta x = (2 - 0)/4 = .5.$

The left endpoints are 0, .5, 1, 1.5; the Riemann sum is

$(0^2 + (.5)^2 + 1^2 + (1.5)^2) \cdot .5 = (.25 + 1 + 2.25).5 = 3.5 \cdot .5 = 1.75.$

The true answer is 8/3=2.666..., so the approximation is not very good. The midpoint method gives better approximations, and we can always improve our approximations by increasing n**.**