7.4. Fundamental Theorem of Calculus—continued

The Fundamental Theorem of Calculus. **Let f be continuous on the interval $[a, b]$, and let F be any antiderivative of f. Then**

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b.$$

Example: $\int_1^2 2x(1+x^2)^3\,dx$ **(let** $u = 1 + x^2$**,** $du = 2x\,dx$**)**

$= \int_2^5 u^3\,du$ **(because** $x = 1 \implies u = 2$ **and** $x = 2 \implies u = 5$**)**

$= \frac{u^4}{4}\Big|_2^5 = \frac{5^4}{4} - \frac{2^4}{4} = \frac{625 - 16}{4} = \frac{609}{4}$**.**

Some properties of definite integrals—assume f and g are continuous **.**

1) $\int_a^a f(x)\,dx = 0$**.**

2) $\int_a^b k \cdot f(x)\,dx = k \int_a^b f(x)\,dx$ **for any constant** k**.**

3) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$**.**

4) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ **for any** c**.**

5) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$**.**

6) Let $F(x) = \int_a^x f(t)\,dt$**. Then** $F'(x) = f(x)$**.**

7.5. Integrals of Trigonometric Functions

$$\int \cos x\,dx = \sin x + C.$$

$$\int \sin x\,dx = -\cos x + C.$$

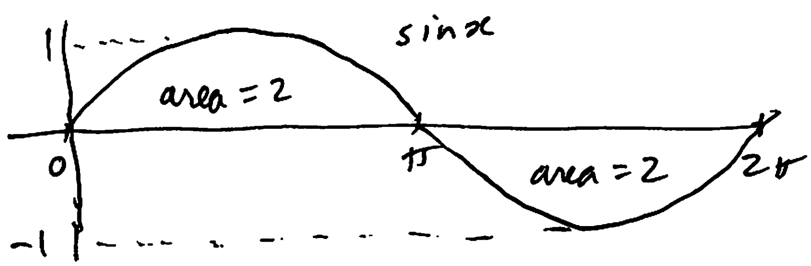
$$\int \sec^2 x\,dx = \tan x + C, \text{ etc.}$$

$$\int x\cos(x^2)\,dx \qquad (\text{let } u = x^2,\ du = 2x\,dx)$$

$$= \int \frac{1}{2}\cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C.$$

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos(\pi) + \cos(0) = 1 + 1 = 2.$$

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = -\cos(2\pi) + \cos(0) = -1 + 1 = 0.$$



$$\int \sin^6 x\cos x\,dx = \sin x + C \qquad (\text{let } u = \sin x,\ du = \cos x\,dx)$$

$$= \int u^6\,du = \frac{u^7}{7} + C = \frac{1}{7}\sin^7 x + C$$

$$\int \frac{\cos x}{1+\sin x}\,dx \qquad \textbf{(let } u = 1+\sin x\textbf{,}\ du = \cos x\,dx\textbf{)}$$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sin x| + C$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad \textbf{(let } u = \cos x\textbf{,}\ du = -\sin x\,dx\textbf{)}$$

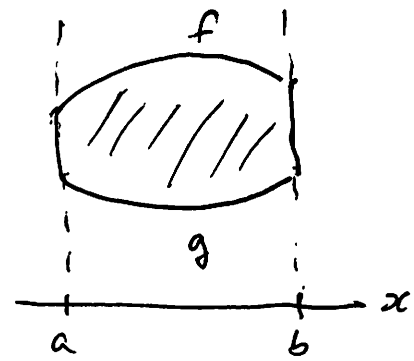
$$= \int -\frac{1}{u}\,du = -\ln|u| + C = -\ln|\sin x| + C\textbf{.}$$

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

$$\textbf{(let } u = \sec x + \tan x\textbf{,}\ du = \sec x(\sec x + \tan x)\,dx\textbf{)}$$

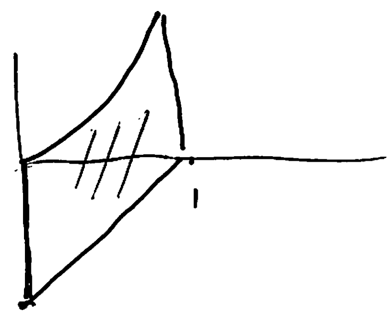
$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sec x + \tan x| + C\textbf{.}$$

7.6. Area between two curves



If $g(x) \le f(x)$ on $[a, b]$, then the area of the region between the graphs of f and g, $a \le x \le b$ is $\int_a^b f(x) - g(x)\,dx$.

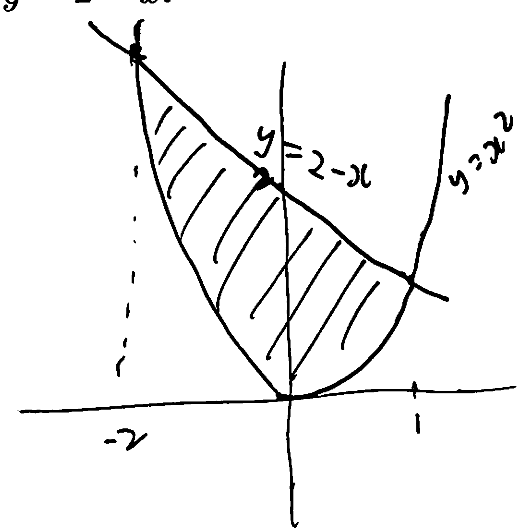
Examples



1. Area of $x - 1 \le y \le x^2$, $0 \le x \le 1$.

$$A = \int_0^1 x^2 - (x-1)\,dx = \frac{x^3}{3} - \frac{x^2}{2} + x\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}.$$

2. Area bounded by the curves $y = x^2$ and $y = 2 - x$.



The curves intersect where $x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1, -2$

$$A = \int_{-1}^1 2 - x - x^2\,dx = 2x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-2}^1 = 4.5$$

3. Area bounded by the curves

$y = f(x) = x^3 - x^2$ **and** $y = g(x) = 2x$.

The curves intersect where $x^3 - x^2 = 2x$

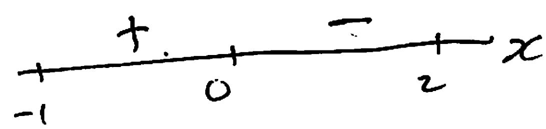
$f(x) - g(x) = x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$

$x(x+1)(x-2) = 0$

$x = -1, 0, 2$

Consider whether $f(x) - g(x)$ **is positive or negative:**



$$A = \int_{-1}^{0} x^3 - x^2 - 2x\,dx + \int_{0}^{2} -(x^3 - x^2 - 2x)\,dx$$