Chapter 11: Differential Equations

One of the most important ways in which Mathematics is used to help understand scientific phenomena is through differential equations.

Differential equations are equations involving derivatives of unknown functions.

The laws of motion of planets, laws of wave motions, vibration of a pendulum, electrical circuits, temperature fluctuation, population growth, spread of disease, are all given by differential equations.

11.1: Solutions of Elementary and Separable Differential Equations

We will study *first order ordinary differential equations*:

$$\frac{dy}{dx} = r(x, y),$$

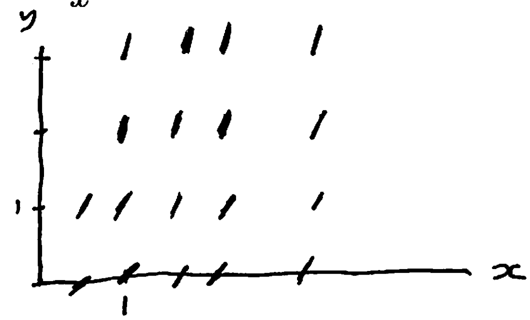
which we will describe as a DE for short.

A *solution* of the DE is a function $y = f(x)$ such that

$$\frac{dy}{dx} = f'(x) = r(x, f(x)).$$

Example: $y = x^2$ **is a solution of** $\frac{dy}{dx} = x + \frac{y}{x}$, $x > 0$.

"slope field"



Generally there is an infinite family of solutions depending on some arbitrary constant. One needs to specify an initial condition (IC) $y(x_0) = y_0$ to determine a specific solution. A problem of solving a DE together with an IC is called an "intitial value problem", or IVP for short. So $y = x^2$ is the solution of the IVP

$$\frac{dy}{dx} = x + \frac{y}{x}, \; y(1) = 1.$$

Separable Equations: $\frac{dy}{dx} = f(x)g(y)$

Solve by separating the variables and integrating:

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$$

3 cases:

1) $\dfrac{dy}{dx} = f(x)$ (**"pure time" if x is time**)

2) $\dfrac{dy}{dx} = g(y)$ (**"time independent" or "autonomous"**)

3) $\dfrac{dy}{dx} = f(x)g(y)$ (**general case**)

Examples: **1) IVP** $\dfrac{dy}{dx} = e^{2x}$, $y_0 = 5$ **when** $x_0 = 0$.

The general solution is $y = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$.

By the IC, $5 = \frac{1}{2}e^0 + C = \frac{1}{2} + C$, **so** $C = 4.5$. **So the solution of the IVP is** $y = \frac{1}{2}e^{2x} + 4.5$.

2) IVP $\dfrac{dy}{dt} = \cos t$, $y(\pi/2) = 0$.

$y = \int \cos t\,dt = \sin t + C$ (**general solution**).

$0 = y(\pi/2) = \sin(\pi/2) + C = 1 + C$, **so** $C = -1$, **so** $y = \sin(t) - 1$.

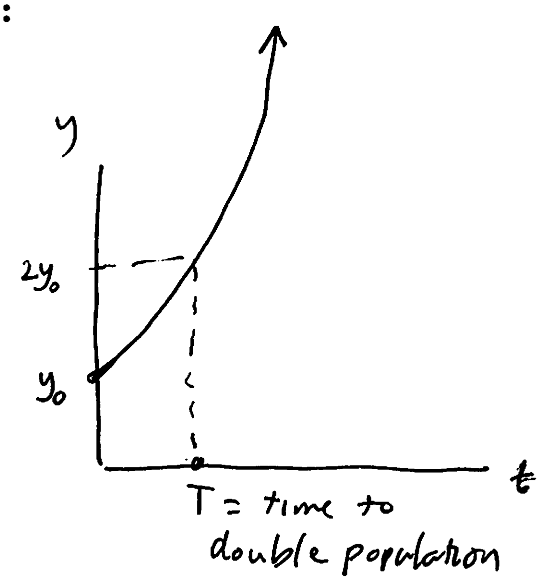
3) The exponential law of growth. Let $y =$ **population at time** t. **Assume that the population grows at a rate proportional to the size of the population, so**

DE: $\frac{dy}{dt} = ry$**, where** r **is a parameter, the "intrinsic rate of growth".**

Let the population at time 0 be y_0 **(the IC).**

Important: the solution is NOT $y = ry^2/2 + C$**.**

We must solve by separating variables:



$$\int \frac{1}{y}\, dy = \int r\, dt$$

$$\ln(|y|) = rt + C$$

$$|y| = e^{\ln(|y|)} = e^{rt+C} = e^{rt}e^{C}$$

$y = \pm e^{C}e^{rt} = Ae^{rt}$ **(general solution)**

$y_0 = y(0) = Ae^0 = A$, **so** $y = y_0 e^{rt}$.

4) $\frac{dy}{dx} = x/y^2$**, (defined for** $x > 0$**),** $y(1) = 2$**.**

Separate variables: $\int y^2\, dy = \int x\, dx$

$y^3/3 = x^2/2 + C$, **so** $y^3 = \frac{3}{2}x^2 + 3C$,

$8 = 2^3 = \frac{3}{2}1^2 = \frac{3}{2} + 3C$, **so** $3C = \frac{13}{2}$

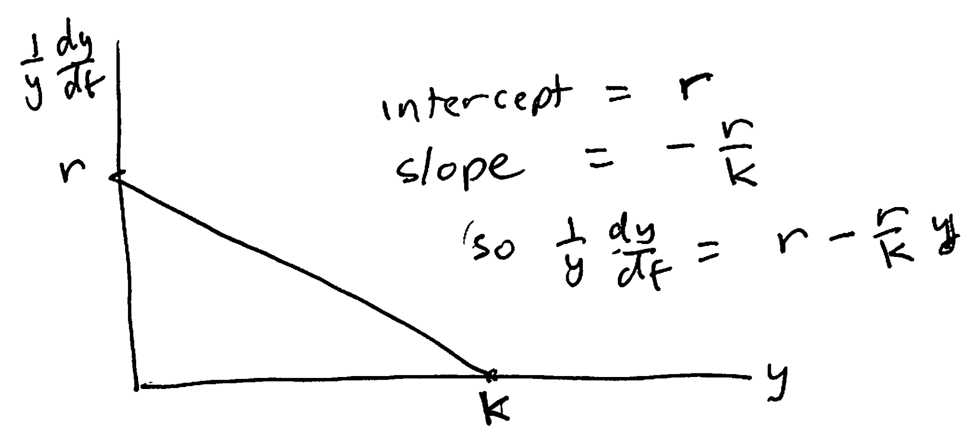
$y = (\frac{3}{2}x^2 + \frac{13}{2})^{1/3}$.

5) The Exponential Law of Growth is a good model for population growth when there are ample resources for the population. When resources are limited, a popular model in Ecology is the "Logistic Law" (due to Verhulst in 1838).

The Exponential DE can be restated as

" per capita growth" $= \frac{1}{y}\frac{dy}{dt} = r.$

Suppose the environment can sustain a maximum population of K**, the "carrying capacity". Then a reasonable model for per capita growth is that it decreases linearly from** r **when** $y = 0$ **to 0 when** $y = K$.



So $\frac{dy}{dt} = ry(1 - \frac{y}{K}) = r\frac{y(K-y)}{K}$ **"Logistic DE"**

Separating variables,

$$rt + C = \int r\,dt = \int \frac{K}{y(K-y)}\,dy = \int \frac{1}{y} + \frac{1}{K-y}\,dy$$

$$= \ln(|y|) - \ln(|K-y|) = \ln(|\frac{y}{K-y}|).$$

$$\frac{y}{K-y} = \pm e^{rt+C} = Ae^{rt}$$

$$y = AKe^{rt} - yAe^{rt}$$

$$y(1 + Ae^{rt}) = AKe^{rt}$$

$y = \dfrac{AKe^{rt}}{1 + Ae^{rt}}$ **and dividing both sides by** Ae^{rt},

$y = \dfrac{K}{1 + be^{-rt}}$, **where** $b = 1/A$.

Solve $y_0 = \dfrac{K}{1+b}$ **to get** $b = \frac{y_0 - K}{y_0}$.