11.2: Linear First Order Differential Equations

A *first order linear ordinary differential equation* **is a DE which can be written as:** $\frac{dy}{dx} + P(x)y = Q(x)$**, where** P **and** Q **can be any functions of** x**.**

Examples: $\frac{dy}{dx} = x^2y + \sin x$ **is linear,**

$\frac{dy}{dx} = y^2$ **and** $\frac{dy}{dx} = \frac{1}{y}$ **are not linear.**

Solve linear DEs by *the method of integrating factors:*

Let $I(x) = e^{\int P(x)\,dx}$**; this is the "integrating factor".**

Then $I'(x) = e^{\int P(x)\,dx} \cdot \frac{d}{dx}\int P(x)\,dx = I(x)P(x)$**.**

Thus $(Iy)' = Iy' + I'y$ **by the Product Rule**

$= Iy' + IPy = I(y' + Py) = IQ$ **if** y **is a solution.**

So $Iy = \int IQ\,dx$**.**

$y(x) = \frac{1}{I(x)}\int I(x)Q(x)\,dx$ **where** $I(x) = e^{\int P(x)\,dx}$.

Examples:

1. $x\dfrac{dy}{dx} - y - x^3 = 0$**,** $x > 0$**.**

Divide by x **and move the last term to the right:** $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2$

So $P = -\dfrac{1}{x}$ **and** $Q = x^2$**.**

$I = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$**.**

$y = x \int \dfrac{1}{x} \cdot x^2\,dx = x \int x\,dx$

$y = x(\frac{1}{2}x^2 + C) = \frac{1}{2}x^3 + Cx$**, for arbitrary constant C.**

Check the answer: $\dfrac{dy}{dx} = \frac{3}{2}x^2 + C$

$x\dfrac{dy}{dx} - y - x^3 = \frac{3}{2}x^3 + Cx - (\frac{1}{2}x^3 + Cx) - x^3 = 0$**.**

2. Find the general solution of $\dfrac{dy}{dx} - 2x = 1$**.**

What are P **and** Q**?**

The integrating factor $I =$

So the solution $y =$

3. Newton's Law of Cooling.

$y =$ **temperature at time** t **of an object placed in a large fluid or gas of constant temperature** A **(the "ambient temperature").**

Assumption: The temperature of the object approaches the ambient temperature at a rate proportional to the temperature difference. This assumption can be restated as

$\frac{dy}{dt} = -k(y - A)$**,** k **a positive constant.**

Let $y(0) = y_0$ **be the starting temperature of the object.**

$\frac{dy}{dt} + ky = kA$**, so** $Q = kA$ **and** $P = k$**, so** $I = e^{\int k\,dt} = e^{kt}$**.**

$y = e^{-kt} \int e^{kt} kA\,dt$

$= e^{-kt}(\frac{1}{k}e^{kt}kA + C) = e^{-kt}(Ae^{kt} + C) = A + Ce^{-kt}$

$y_0 = y(0) = A + C$**, so** $C = y_0 - A$**, so** $y(t) = A + (y_0 - A)e^{-kt} \to A$ **as** $t \to 0$**.**

In 1938 von Bertalanffy used the same differential equation to describe the growth of the length of animals, particularly fish, and it is the most popular such model (A represents the full-grown length).

4. Exponential growth with immigration/emigration. Suppose $Q(t) =$ rate of immigration $-$ rate of emigration . Then we have the model:

$\frac{dy}{dt} = ky + Q(t)$**,** $k > 0$**. Rewrite the DE as** $\frac{dy}{dt} - ky = q(t)$**.**

So $P(t) = -k$ **and** $I(t) = e^{\int -k\,dt} = e^{-kt}$**.**

The solution is $y = e^{kt} \int e^{-kt} Q(t)\,dt$**.**

Suppose the $Q(t) = q$ **constant (maybe positive or negative).**

Then $y = e^{kt} \int e^{-kt} q\,dt = e^{kt}(\frac{1}{-k} e^{-kt} q) + C$

$y = -\frac{q}{k} + Ce^{kt}$ **and** $y_0 = y(0) = -\frac{q}{k} + C$**, so** $C = y_0 + \frac{q}{k}$**.**

The population grows to infinity if $C > 0$**, that is** $y_0 > -\frac{q}{k}$ **or** $y_0 k > -q$**; the population goes to zero if** $y_0 k < -q$**.**