Chapter 3 The derivative

3.1 Limits

\[ \lim_{x \to a} f(x) = l \] means roughly “as \( x \) approaches \( a \) (but not necessarily when \( x = a \)), \( f(x) \) is approaching \( l \).” We don’t require \( f \) to be defined at \( a \).

Let \( f(x) = \frac{x^2 - 4}{x - 2} \), domain \( x \neq 2 \). Table:

\[
\begin{array}{c|c|c}
\hline
x & f(x) & x \\
\hline
3 & 5 & 1.9 \\
2.1 & 4.1 & 3.9 \\
2.01 & 4.01 & 3.99 \\
2.001 & 4.001 & 3.999 \\
\hline
\end{array}
\]

Let \( g(x) = \frac{\sin x}{x} \), Table:

\[
\begin{array}{c|c}
\hline
x & g(x) \\
\hline
\pm 0.1 & 0.998334... \\
\pm 0.01 & 0.9998333... \\
\pm 0.001 & 0.99998333... \\
\hline
\end{array}
\]

"Limits by calculator method"

We can define one-sided limits in which we only consider \( x \) approaching \( a \) from one side; this is written \( \lim_{x \to a^-} f(x) \) if we let \( x \) approach \( a \) from the left (only considering numbers \( x < a \)) and \( \lim_{x \to a^+} f(x) \) if we let \( x \) approach \( a \) from the left (only considering numbers \( x > a \)).
\[ \lim_{x \to 0^+} \frac{|x|}{x} = 1, \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1, \quad \text{but} \quad \lim_{x \to 0} \frac{|x|}{x} \text{ does not exist.} \]

\[ \frac{|x|}{x} = \frac{x}{x} = 1 \quad \text{if} \quad x > 0, \quad \frac{|x|}{x} = \frac{-x}{x} = -1 \quad \text{if} \quad x < 0 \]

\[ \lim_{x \to 2^+} \frac{1}{x-2} = \infty, \quad \lim_{x \to 2^-} \frac{1}{x-2} = -\infty, \quad \text{and} \quad \lim_{x \to 2} \frac{1}{x-2} = \pm \infty. \]

<table>
<thead>
<tr>
<th>\frac{x}{x-2}</th>
<th>\frac{1}{x-2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>0</td>
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<tr>
<td>2.01</td>
<td>0.01</td>
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<tr>
<td>2.001</td>
<td>0.001</td>
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<tr>
<td>...</td>
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</tbody>
</table>

It is also correct for any of these limits to say “does not exist”, but the \( \infty \) is more informative, because there are various ways for a limit not to exist.

\[ \lim_{x \to 0^+} \sin \frac{1}{x} \text{ does not exist:} \]

Let \( f(x) = \begin{cases} 0, & \text{for } x \text{ rational} \\ 1, & \text{for } x \text{ irrational} \end{cases} \).

The limit does not exist anywhere.
Formal definition: Suppose $a, L$ are constants and $f$ is a function defined near $a$. Suppose for every $\epsilon > 0$ there corresponds a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

We won't ever use this, but it can be used to prove the following important properties of limits (see p140 of the text):

Suppose $a$ and $k$ are constants, and $f$ and $g$ are functions defined near $a$. Assume $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} f(x) = A$. Then:

1) (constant property) $\lim_{x \to a} k f(x) = k A$;
2) (sum property) $\lim_{x \to a} f(x) + g(x) = A + B$;
3) (difference property) $\lim_{x \to a} f(x) - g(x) = A - B$;
4) (product property) $\lim_{x \to a} f(x) \cdot g(x) = A \cdot B$;
5) (quotient property) $\lim_{x \to a} f(x)/g(x) = A/B$ if $B \neq 0$.

It is easy from the definition to see that $\lim_{x \to a} x = a$.

Then from the product property $\lim_{x \to a} x^2 = a^2$.

Similarly $\lim_{x \to a} x^3 = \lim_{x \to a} x^2 \cdot \lim_{x \to a} x = a^2 a = a^3$.

In general for any positive integer $n$, $\lim_{x \to a} x^n = a^n$.

Using properties 1)-4), $\lim_{x \to a} 3x^2 + 4x - 5 = 3a^2 + 4a - 5$.

In this way we can find all limits of any polynomial.
Limits of rational functions:
\[ \lim_{x \to 2} \frac{x^2 - 4}{x + 1} = \frac{0}{3} = 0. \]

\[ \lim_{x \to 2} \frac{x^2 + 4}{x - 2} = \frac{8}{0} = \infty \text{ (or: "does not exist")}. \]

\[ \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0} = ? \]

\[ = \lim_{x \to 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} x + 2 = 4. \]

The method of these three examples works for all rational functions.

"Pieced together functions". Let

\[ f(x) = \begin{cases} 
   x + 1 & x < 1 \\
   x^2 - 1 & x > 1
\end{cases} \]

\[ \lim_{x \to a} f(x) = a + 1 \text{ if } a < 1 \text{ and } \lim_{x \to a} f(x) = a^2 - 1 \text{ if } a > 1; \]

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1} x + 1 = 2, \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1} x^2 - 1 = 0; \]

since the left- and right-hand limits are unequal, \( \lim_{x \to 1} f(x) \) does not exist.
Limits at ∞

\[ \lim_{x \to \infty} f(x) = l \] means that \( f(x) \) approaches \( l \) as \( x \) gets arbitrarily large.

In this case the graph of \( f \) has a horizontal asymptote \( y = l \).

The calculations of these limits for rational functions is fairly easy:

Note that \( \lim_{x \to \infty} \frac{1}{x} = 0 \), so \( \lim_{x \to \infty} \frac{1}{x^n} = 0 \) for any positive integer \( n \) (actually this holds for any real \( n > 0 \)).

\[
\begin{align*}
\lim_{x \to \infty} \frac{2x^2 + x + 1}{3x^2 + 4} &= \lim_{x \to \infty} \frac{2 + 1/x + 1/x^2}{3 + 4/x^2} = \frac{2}{3} \\
\lim_{x \to \infty} \frac{x + 7}{2x^2 + 1} &= \lim_{x \to \infty} \frac{1/x^2 + 7/x^3}{2 + 1/x^3} = \frac{0 + 0}{2 + 0} = 0
\end{align*}
\]

Try this one: \( \lim_{x \to \infty} \frac{x^3}{2x^2 + 1} = ? \)

Answer = \( \infty \).