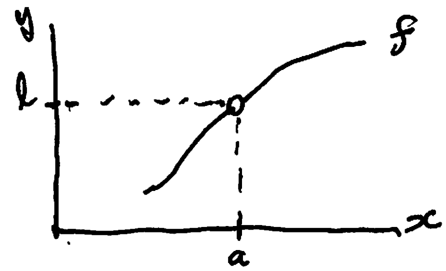
Chapter 3 The derivative

3.1 Limits

$\lim_{x \to a} f(x) = l$ means roughly "as x approaches a (but not necessarily when $x = a$), $f(x)$ is approaching l". We don't require f to be defined at a.



Let $f(x) = \dfrac{x^2 - 4}{x - 2}$, domain $x \neq 2$. Table:

$\lim_{x \to 2} f(x) = 4$

x	$f(x)$
3	5
2.1	4.1
2.01	4.01
2.001	4.001
⋮	⋮

x	$f(x)$
1.9	3.9
1.99	3.99
1.999	3.999
⋮	⋮

Let $g(x) = \dfrac{\sin x}{x}$, Table:

$\lim_{x \to 0} g(x) = 1.$

x	$g(x)$
±.1	.998334...
±.01	.99998333...
±.001	.9999998333...

"Limits by calculator method"

We can define one-sided limits in which we only consider x approaching a from one side; this is written $\lim_{x \to a^-} f(x)$ if we let x approach a from the left (only considering numbers $x < a$) and $\lim_{x \to a^+} f(x)$ if we let x approach a from the left (only considering numbers $x > a$).

$\lim\limits_{x\to 0+} \frac{|x|}{x} = 1$, $\lim\limits_{x\to 0-} \frac{|x|}{x} = -1$, **but** $\lim\limits_{x\to 0} \frac{|x|}{x}$ **does not exist.**

$\frac{|x|}{x} = \frac{x}{x} = 1$ if $x > 0$, $\frac{|x|}{x} = \frac{-x}{x} = -1$ if $x < 0$

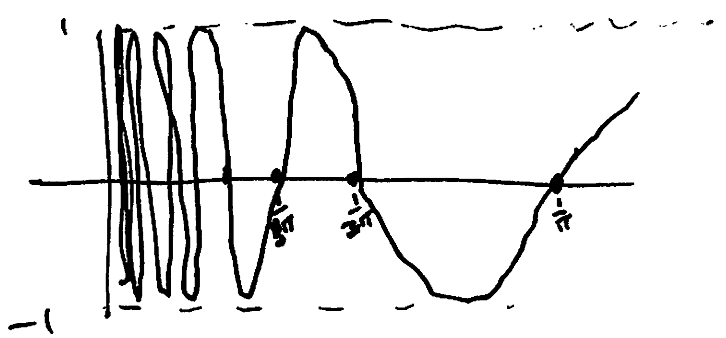
$\lim\limits_{x\to 2+} \frac{1}{x-2} = \infty$, $\lim\limits_{x\to 2-} \frac{1}{x-2} = -\infty$, **and** $\lim\limits_{x\to 2} \frac{1}{x-2} = \pm\infty$.

x	$\frac{1}{x-2}$
2.1	10
2.01	100
2.001	1000
⋮	

x	$\frac{1}{x-2}$
1.9	−10
1.99	−100
1.999	−1000
⋮	

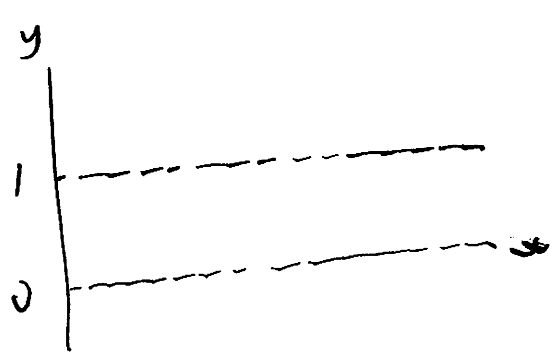
It is also correct for any of these limits to say "does not exist", but the ∞ is more informative, because there are various ways for a limit not to exist.

$\lim\limits_{x\to 0+} \sin\frac{1}{x}$ **does not exist:**

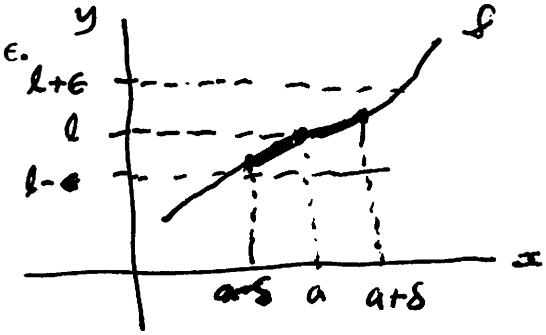


Let $f(x) = \begin{cases} 0, & \text{for } x \text{ rational} \\ 1, & \text{for } x \text{ irrational} \end{cases}$.

The limit does not exist anywhere.



Formal definition: Suppose a, L are constants and f is a function defined near a. Suppose for every $\epsilon > 0$ there corresponds a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \epsilon$.



We won't ever use this, but it can be used to prove the following important properties of limits (see p140 of the text):

Suppose a and k are constants, and f and g are functions defined near a. Assume $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} f(x) = A$. Then:

1) (constant property) $\lim_{x \to a} kf(x) = kA$;

2) (sum property) $\lim_{x \to a} f(x) + g(x) = A + B$;

3) (difference property) $\lim_{x \to a} f(x) - g(x) = A - B$;

4) (product property) $\lim_{x \to a} f(x) \cdot g(x) = A \cdot B$;

5) (quotient property) $\lim_{x \to a} f(x)/g(x) = A/B \quad$ if $B \neq 0$.

It is easy from the definition to see that $\lim_{x \to a} x = a$.

Then from the product property $\lim_{x \to a} x^2 = a^2$.

Similarly $\lim_{x \to a} x^3 = \lim_{x \to a} x^2 \cdot \lim_{x \to a} x = a^2 a = a^3$.

In general for any positive integer n, $\lim_{x \to a} x^n = a^n$.

Using properties 1)-4), $\lim_{x \to a} 3x^2 + 4x - 5 = 3a^2 + 4a - 5$.

In this way we can find all limits of any polynomial.

Limits of rational functions:

$$\lim_{x\to 2} \frac{x^2-4}{x+1} = \frac{0}{3} = 0.$$

$$\lim_{x\to 2} \frac{x^2+4}{x-2} = \frac{8}{0} = \infty \text{ (or: “does not exist”)}$$

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{0}{0} = ?$$

$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 4.$$

The method of these three examples works for all rational functions.

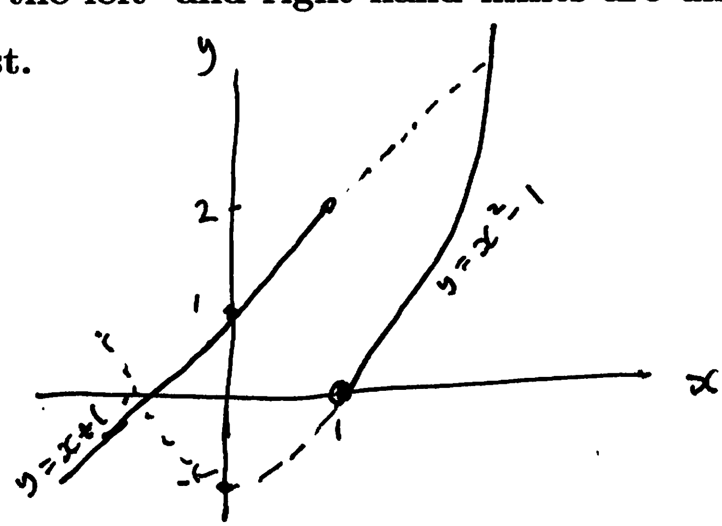
“Pieced together functions”. Let

$$f(x) = \begin{cases} x+1 & x<1 \\ x^2-1 & x>1 \end{cases}$$

$\lim_{x\to a} f(x) = a+1$ **if** $a<1$ **and** $\lim_{x\to a} f(x) = a^2-1$ **if** $a>1$;

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x+1 = 2$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1} x^2-1 = 0$;

since the left- and right-hand limits are unequal, $\lim_{x\to 1} f(x)$ **does not exist.**



Limits at ∞

$\lim_{x\to\infty} f(x) = l$ **means that** $f(x)$ **approaches** l **as** x **gets arbitrarily large.**

In this case the graph of f **has a horizontal asymptote** $y = l$**.**

The calculations of these limits for rational functions is fairly easy:

Note that $\lim_{x\to\infty} \frac{1}{x} = 0$**, so** $\lim_{x\to\infty} \frac{1}{x^n} = 0$ **for any positive integer (actually this holds for any real** $n > 0$**.**

$$\lim_{x\to\infty} \frac{2x^2+x+1}{3x^2+4} = \lim_{x\to\infty} \frac{2+1/x+1/x^2}{3+4/x^2} = \frac{2}{3}$$

$$\lim_{x\to\infty} \frac{x+7}{2x^2+1} = \lim_{x\to\infty} \frac{1/x^2+7/x^3}{2+1/x^3} = \frac{0+0}{2+0} = 0$$

Try this one: $\lim_{x\to\infty} \frac{x^3}{2x^2+1} =?$

Answer $= \infty$**.**