3.3 Rates of Change—continued

Suppose variable $y$ is a function of variable $x$: $y = f(x)$.

The instantaneous rate of change $y$ with respect to $x$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

The instantaneous rate of change of position $s$ with respect to time $t$ is called instantaneous velocity (or just velocity).

Eg # 10, p.169: An object moving on a line has position $s = f(t) = t^2 + 5t + 2$. Find the velocity at $t = 1$.

The average velocity on $1 \leq t \leq 1 + h$ is

$$\frac{f(1 + h) - f(1)}{h} = \frac{(1 + h)^2 + 5(1 + h) + 2 - (1 + 5 + 2)}{h} = \frac{1 + 2h + h^2 + 5 + 5h + 2 - 8}{h} = \frac{7h + h^2}{h} = 7 + h.$$

The instantaneous velocity is the limit as $h$ goes to 0, that is 7.
3.4 Derivative

We want to find the slope of a curve $y = f(x)$ at $(a, f(a))$. The line joining $(a, f(a))$ and $(a + h, f(a + h))$ is called a secant line; its slope is

$$\frac{f(a + h) - f(a)}{h}$$

The slope of the tangent line at $(a, f(a))$ is the limit of these slopes as $h$ goes to 0. We call this the derivative of $f$ at $a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$ 

Note it is the instantaneous rate of change of $y = f(x)$ at $x = a$.

$f(x) = 5x + 3$.

$$f'(a) = \lim_{h \to 0} \frac{5(a + h) + 3 - (5a + 3)}{h} = \lim_{h \to 0} \frac{5h}{h} = 5.$$

More generally, for $f(x) = mx + b$, $f'(x) = m$.

$f(x) = x^2$.

$$f'(a) = \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h} = \lim_{h \to 0} \frac{2ah + h^2}{h} = \lim_{h \to 0} 2a + h = 2a.$$

The tangent line to $y = x^2$ at $(3,9)$ is $y - 9 = f'(3)(x - 3) = 6(x - 3)$. 
\[ f' \text{ is another function. It's convenient to use variable } x \text{ again:} \]

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \]

\[ f(x) = \frac{1}{x}. \]

\[ f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{x - (x + h)}{x(x + h)} = \lim_{h \to 0} \frac{-1}{x(x + h)} = -\frac{1}{x^2}. \]

\[ g(x) = \sqrt{x}, \ x \geq 0. \]

\[ g'(x) = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} = \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})} = \frac{1}{2\sqrt{x}} \]

for \( x > 0 \) \((g'(0) \text{ does not exist}).\)

\[ h(x) = |x| = \begin{cases} 
    x & x \geq 0 \\
    -x & x < 0
\end{cases} \]

\[ h'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \begin{cases} 
    1 & x > 0 \\
    -1 & x < 0
\end{cases} \]

but

\[ h'(0) = \lim_{h \to 0} \frac{|h|}{h}. \]

The right limit is 1 and the left limit is \(-1\), so the limit does not exist, so \(h'(0)\) does not exist.
$f'$ can be undefined in other ways:

$f(x) = x^{1/3}$ looks like:

$g(x) = x^{2/3}$ looks like:

Both of these have infinite slope at $x = 0$.

More precisely, $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to 0^+} g(x) = \infty$ but $\lim_{x \to 0^-} g(x) = -\infty$.

$f$ is said to be differentiable at $a$ if $f'(a)$ exists.

Theorem. If $f$ is differentiable at $a$ then $f$ is continuous at $a$; equivalently, if $f$ is not continuous at $a$ then $f$ is not differentiable at $a$.

So $f$ is not differentiable at $a$ in the following cases: