Suppose variable y is a function of variable x: $y = f(x)$.

The *instantaneous rate of change y with respect to x at $x = a$* **is**

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The instantaneous rate of change of position s with respect to time t is called *instantaneous velocity* **(or just velocity).**

Eg # 10, p.169: An object moving on a line has position $s = f(t) = t^2 + 5t + 2$. Find the velocity at $t = 1$.

The average velocity on $1 \leq t \leq 1 + h$ is

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 + 5(1+h) + 2 - (1+5+2)}{h} =$$

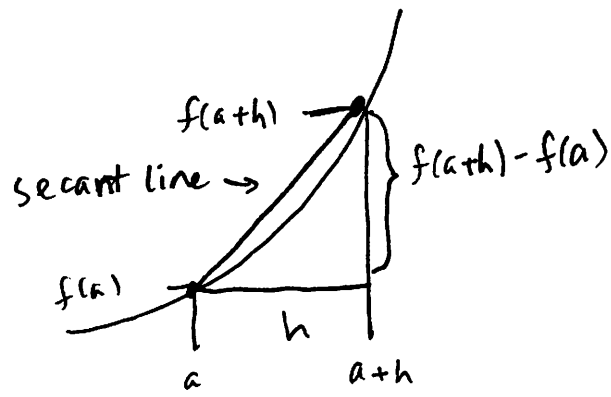
$$\frac{1 + 2h + h^2 + 5 + 5h + 2 - 8}{h} = \frac{7h + h^2}{h} = 7 + h.$$

The instantaneous velocity is the limit as h goes to 0, that is 7.

3.4 Derivative

We want to find the slope of a curve $y = f(x)$ at $(a, f(a))$. The line joining $(a, f(a))$ and $(a+h, f(a+h))$ is called a *secant line*; its slope is

$$\frac{f(a+h) - f(a)}{h}.$$



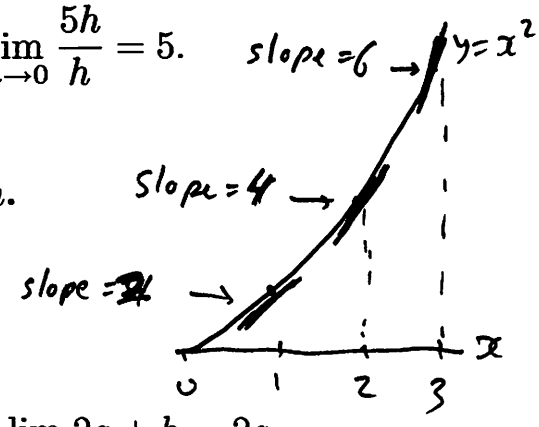
The slope of the tangent line at $(a, f(a))$ is the limit of these slopes as h goes to 0. We call this the *derivative of f at a*:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Note it is the instantaneous rate of change of $y = f(x)$ at $x = a$.

$f(x) = 5x + 3.$

$$f'(a) = \lim_{h \to 0} \frac{5(a+h) + 3 - (5a+3)}{h} = \lim_{h \to 0} \frac{5h}{h} = 5.$$



More generally, for $f(x) = mx + b$, $f'(x) = m$.

$f(x) = x^2.$

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{2ah + h^2}{h} = \lim_{h \to 0} 2a + h = 2a.$$

The tangent line to $y = x^2$ at (3,9) is $y - 9 = f'(3)(x-3) = 6(x-3)$.

f' is another function. It's convenient to use variable x again:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

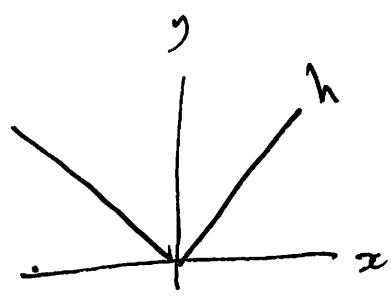
$f(x) = \dfrac{1}{x}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\frac{x-(x+h)}{x(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

$g(x) = \sqrt{x}$, $x \geq 0$.

$$g'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

for $x > 0$ ($g'(0)$ does not exist).



$$h(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$h'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

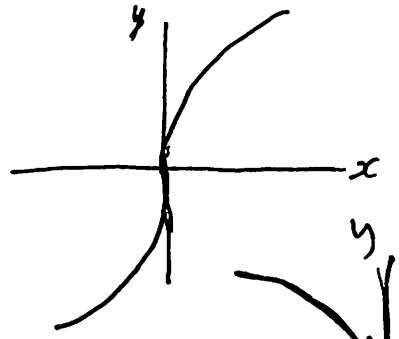
but

$$h'(0) = \lim_{h\to 0} \frac{|h|}{h}.$$

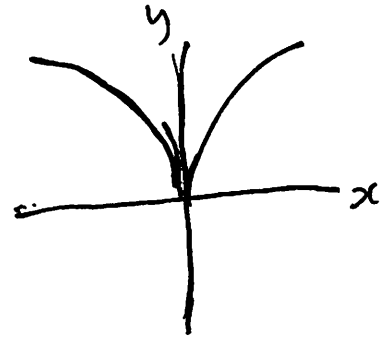
The right limit is 1 and the left limit is -1, so the limit does not exist, so $h'(0)$ does not exist.

f' can be undefined in other ways:

$f(x) = x^{1/3}$ looks like:



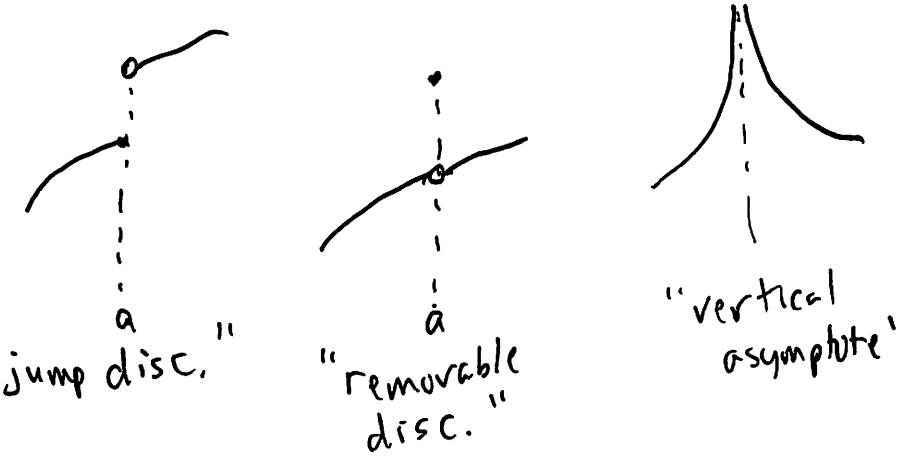
$g(x) = x^{2/3}$ looks like:



Both of these have infinite slope at $x = 0$.

More precisely, $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to 0^+} g(x) = \infty$ but $\lim_{x \to 0^-} g(x) = -\infty$.

f is said to be *differentiable* at a if $f'(a)$ exists.

Theorem. If f is differentiable at a then f is continuous at a; equivalently, if f is not continuous at a then f is not differentiable at a.

So f is not differentiable at a in the following cases: