4.2 Derivatives of products and quotients—continued.

Review of rules:

Constant rule: \( \frac{d}{dx} ku(x) = ku'(x) \).

Sum/Difference rules: \( \frac{d}{dx} u(x) \pm v(x) = u'(x) \pm v'(x) \).

Product rule: \( \frac{d}{dx} u(x)v(x) = u(x)v'(x) + u'(x)v(x) \).

Reciprocal rule: \( \frac{d}{dx} \frac{1}{v(x)} = \frac{-v'(x)}{v(x)^2} \).

Quotient rule: \( \frac{d}{dx} \frac{u(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2} \).

Proof of product rule.

If \( f(x) = u(x)v(x) \), then \( f'(x) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - (u(x)v(x))}{h} = \)

\[ \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} = \]

\[ \lim_{h \to 0} \left[ u(x+h) \left( \frac{v(x+h) - v(x)}{h} \right) + \left( \frac{u(x+h) - u(x)}{h} \right) v(x) \right] = \]

\[ \lim_{h \to 0} u(x+h) \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} + \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} v(x) = \]

\( u(x)v'(x) + u'(x)v(x) \).
4.3 The Chain Rule.

Composition: \((g \circ f)(x) = g(f(x))\).

Example: \(f(x) = x^2 - 1\), \(g(x) = \sqrt[3]{x} = x^{1/3}\);

\((f \circ g)(x) = f(g(x)) = (\sqrt[3]{x})^2 - 1 = x^{2/3} - 1\) and \((g \circ f)(x) = \sqrt[3]{x^2} - 1\)

Then \((f \circ g)(3) = 3^{2/3} - 1 = 1.0801\), but \((g \circ f)(3) = \sqrt[3]{3^2} - 1 = \sqrt[3]{8} = 2\).

Let \(f\) and \(g\) be differentiable functions.

Consider variables \(y = g(x), z = f(y) = f(g(x)), k = g(x + h) - g(x)\);

note that \(g(x + h) = y + k\) and \(k \to 0\) as \(h \to 0\) because \(g\) is continuous.

\[
(f \circ g)'(x) = \lim_{h \to 0} \frac{(f \circ g)(x + h) - (f \circ g)(x)}{h} \\
= \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{h} \\
= \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)} \cdot \frac{g(x + h) - g(x)}{h} \\
= \lim_{k \to 0} \frac{f(y + k) - f(y)}{k} \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \\
= f'(g(x))g'(x)
\]

This is the Chain Rule: \((f \circ g)'(x) = f'(g(x))g'(x)\).

“The derivative of a composition is the product of the derivatives, but with the derivative of the outer function \(f\) evaluated at the value of the inner function \(g\).”
Chain Rule: \((f \circ g)'(x) = f'(g(x))g'(x)\).

Examples.

\[
\frac{d}{dx} (x^2 - 1)^{1/3} = \frac{1}{3} (x^2 - 1)^{-2/3} \frac{d}{dx} (x^2 - 1) = \frac{1}{3} (x^2 - 1)^{-2/3} 2x
\]

More generally, \(\frac{d}{dx} (g(x))^n = n(g(x))^{n-1} g'(x)\).

\[
\frac{d}{dx} \left( \frac{1}{x^3 + x + 1} \right) = \frac{d}{dx} ((x^3 + x + 1)^{-1}) = \\
-(x^3 + x + 1)^{-2}(3x^2 + 1) = \frac{3x^2 + 1}{(x^3 + x + 1)^2}.
\]

More generally, \(\frac{d}{dx} \left( \frac{1}{g(x)} \right) = \frac{d}{dx} g(x)^{-1} = -g(x)^{-2} g'(x) = -\frac{g'(x)}{g(x)^2}\),
which is the Reciprocal Rule.

Instead of memorizing the Quotient Rule, you can derive it from the Chain Rule and Product Rule:

\[
\left( \frac{f(x)}{g(x)} \right)' = (g(x)^{-1} f(x))' = g(x)^{-1} f'(x) + (g(x)^{-1})' f(x) = \\
g(x) g(x)^{-2} f'(x) - g(x)^{-2} g'(x) f(x) = \frac{g(x) f'(x) - g'(x) f(x)}{g(x)^2}
\]
The Chain Rule in Leibniz notation:

Let $z = f(y)$, $y = g(x)$, so $z = f(g(x))$.

$$
\frac{dz}{dx} = (f(g(x)))' = f'(g(x))g'(x) = \frac{dz}{dy} \frac{dy}{dx}.
$$

“If $z$ changes $a$ times as fast as $y$, and $y$ changes $b$ times as fast as $x$, then $z$ changes $ab$ times as fast as $x$.”

Textbooks problem 55, p.233: $L =$ length, $w =$ weight, $t =$ time.

For the African wild dog, $L = 2.472w^{2.571}$, (length in mm, weight in kg). For a dog less then one year old, the weight is estimated by $w = .265 + .21t$ (linear growth; $t$ in weeks). How fast is the length of a 25-week-old dog changing?

We want $\frac{dL}{dt}$ when $t = 25$.

At this time, we have $w = .265 + .21 \cdot 25 = 5.515$.

Chain Rule: $\frac{dL}{dt} = \frac{dL}{dw} \frac{dw}{dt} =$

$(2.472)(2.571)w^{1.571} \cdot .21 = 1.335(5.515)^{1.571} = 19.52 \text{ mm/wk}$