4.2 Derivatives of products and quotients—continued.

Review of rules:

Constant rule: $\frac{d}{dx}ku(x) = ku'(x)$**.**

Sum/Difference rules: $\frac{d}{dx}u(x) \pm v(x) = u'(x) \pm v'(x)$**.**

Product rule: $\frac{d}{dx}u(x)v(x) = u(x)v'(x) + u'(x)v(x)$**.**

Reciprocal rule: $\frac{d}{dx}\frac{1}{v(x)} = \frac{-v'(x)}{v(x)^2}$**.**

Quotient rule: $\frac{d}{dx}\frac{u(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2}$**.**

Proof of product rule.

If $f(x) = u(x)v(x)$**, then** $f'(x) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - (u(x)v(x))}{h} =$

$$\lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x))}{h} =$$

$$\lim_{h\to 0} \left[u(x+h)\left(\frac{v(x+h) - v(x)}{h}\right) + \left(\frac{u(x+h) - u(x)}{h}v(x)\right)\right] =$$

$$\lim_{h\to 0} u(x+h) \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + \lim_{h\to 0} \frac{u(x+h) - u(x)}{h}v(x) =$$

$u(x)v'(x) + u'(x)v(x)$**.**

4.3 The Chain Rule.

Composition: $(g \circ f)(x) = g(f(x))$**.**

Example: $f(x) = x^2 - 1$**,** $g(x) = \sqrt[3]{x} = x^{1/3}$**;**

$(f \circ g)(x) = f(g(x)) = (\sqrt[3]{x})^2 - 1 = x^{2/3} - 1$ **and** $(g \circ f)(x) = \sqrt[3]{x^2 - 1}$

Then $(f \circ g)(3) = 3^{2/3} - 1 = 1.0801$**, but** $(g \circ f)(3) = \sqrt[3]{3^2 - 1} = \sqrt[3]{8} = 2$**.**

Let f **and** g **be differentiable functions.**

Consider variables $y = g(x)$**,** $z = f(y) = f(g(x))$**,** $k = g(x+h) - g(x)$**; note that** $g(x+h) = y + k$ **and** $k \to 0$ **as** $h \to 0$ **because** g **is continuous.**

$$
\begin{aligned}
(f \circ g)'(x) &= \lim_{h \to 0} \frac{(f \circ g)(x+h) - (f \circ g)(x)}{h} \\
&= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{k \to 0} \frac{f(y+k) - f(y)}{k} \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(g(x))g'(x)
\end{aligned}
$$

This is the Chain Rule: $(f \circ g)'(x) = f'(g(x))g'(x)$**.**

"The derivative of a composition is the product of the derivatives, but with the derivative of the outer function f **evaluated at the value of the inner function** g**."**

Chain Rule: $(f \circ g)'(x) = f'(g(x))g'(x)$.

Examples.

$$\frac{d}{dx}(x^2-1)^{1/3} = \tfrac{1}{3}(x^2-1)^{-2/3}\frac{d}{dx}(x^2-1) = \tfrac{1}{3}(x^2-1)^{-2/3}2x$$

More generally, $\dfrac{d}{dx}(g(x))^n = n(g(x))^{n-1}g'(x)$.

$$\frac{d}{dx}\left(\frac{1}{x^3+x+1}\right) = \frac{d}{dx}((x^3+x+1)^{-1}) =$$

$$-(x^3+x+1)^{-2}(3x^2+1) = \frac{3x^2+1}{(x^3+x+1)^2}.$$

More generally, $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = \dfrac{d}{dx}g(x)^{-1} = -g(x)^{-2}g'(x) = -\dfrac{g'(x)}{g(x)^2}$, **which is the Reciprocal Rule.**

Instead of memorizing the Quotient Rule, you can derive it from the Chain Rule and Product Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = (g(x)^{-1}f(x))' = g(x)^{-1}f'(x) + (g(x)^{-1})'f(x) =$$

$$g(x)g(x)^{-2}f'(x) - g(x)^{-2}g'(x)f(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g(x)^2}$$

The Chain Rule in Leibniz notation:

Let $z = f(y)$**,** $y = g(x)$**, so** $z = f(g(x))$**.**

$$\frac{dz}{dx} = (f(g(x)))' = f'(g(x))g'(x) = \frac{dz}{dy}\frac{dy}{dx}.$$

"If z **changes** a **times as fast as** y**, and** y **changes** b **times as fast as** x**, then** z **changes** ab **times as fast as** x**."**

Textbooks problem 55, p.233: $L =$ **length,** $w =$ **weight,** $t =$ **time. For the African wild dog,** $L = 2.472w^{2.571}$**, (length in mm, weight in kg). For a dog less then one year old, the weight is estimated by** $w = .265 + .21t$ **(linear growth;** t **in weeks). How fast is the length of a 25-week-old dog changing?**

We want $\frac{dL}{dt}$ **when** $t = 25$**.**

At this time, we have $w = .265 + .21 \cdot 25 = 5.515$**.**

Chain Rule: $\frac{dL}{dt} = \frac{dL}{dw}\frac{dw}{dt} =$

$(2.472)(2.571)w^{1.571} \cdot .21 = 1.335(5.515)^{1.571} = 19.52$ **mm/wk**