4.4 The exponential function.

a^r **base** a **exponent (or power)** r

Meaning? if r **is a positive integer,** a **is multiplied by itself** r **times:** $a^3 = a \cdot a \cdot a$.

Main properties: $a^m a^n = a^{m+n}$ **and** $a^{mn} = a^{mn}$

$a^{-n} = \frac{1}{a^n}$, $3^{-2} = \frac{1}{x^2} = \frac{1}{9}$.

Define $a^0 = 1$.

Fractional r: $a^{1/3} =$ **cube root of** a; **so** $(a^{1/3})^3 = a$.

$4^{1/2} = 2$, **not** -2; $a^{1/\text{even}}$ **is only defined for** $a > 0$.

$8^{5/3} = (8^{1/3})^5 = 2^5 = 32$

Using limits it is possible to define a^r **for** r **an irrational number:**

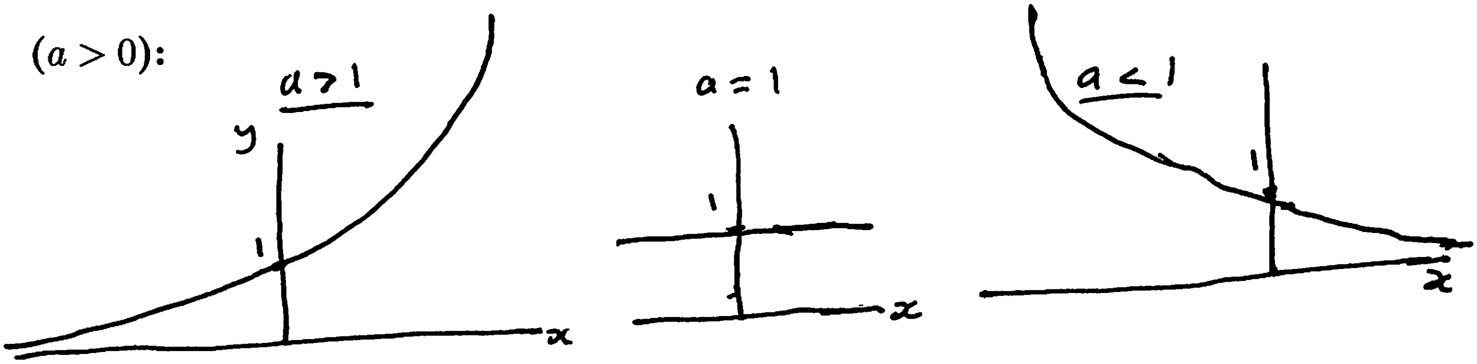
$\pi = 3.1416\ldots$ **so 3, 3.1, 3.14, 3.141, 3.1416 etc. approach** π

2^3, $2^{3.1}$, $2^{3.14}$, $2^{3.141}$, $2^{3.1416}$ **etc. approach** 2^π

If we vary the base and fix the power, eg $f(x) = x^2$ **we get a "power function"**

If we vary the power and fix the base, eg $f(x) = 2^x$ **we get an "exponential function"**

Graphs and properties of an exponential functions $f(x) = a^x$, $(a > 0)$:



$f(0) = 1$; if T is a number so that $f(T) = 2$, so f doubles in time T, then $f(nT) = a^{nT} = (a^T)^n = 2^n$, so f keeps doubling every time interval T.

"exponential growth" if $a > 1$; "exponential decay" if $0 < a < 1$

The most important base is e (Euler's number)

$(1 + \frac{1}{m})^m \to e$ **as** $m \to \infty$**; compute for** $m = 1, 2, 3, 10, 100, 1000$:

$2, 1.5^2 = 2.25, 1.33^3 = 2.37, 1.1^{10} = 2.593, 1.01^{100} = 2.7048, 1.001^{1000} = 2.7169$

if you keep calculating you get $e = 2.718\ldots$

Derivative of exponential functions

Let $f(x) = e^x = \exp(x)$, the "natural exponential function".

$$f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h} = ?$$

h	$(e^h - 1)/h$
.1	1.0517...
.01	1.005017...
.001	1.00050017...
.0001	1.0000500017...
⋮	⋮

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x$$

Practice: $\frac{d}{dx} e^{x^2} =$

$e^{x^2} \frac{d}{dx} x^2 = e^{x^2} 2x$

$\frac{d}{dx} xe^x =$

$xe^x + 1 \cdot e^x = (x+1)e^x$.

Law of Exponential Growth.

Let $P(t)$ **= pop. at time** t; $P_0 = P(0)$ **= initial pop.**

Assume the rate of growth of the pop. is proportional to the size of the pop.; so $P' = kP$ **for some constant** $k > 0$. $P(0) = P_0 =$ **the initial population**

Find a function $P(t)$ **which has this property.**

Let $P(t) = Ce^{kt}$. **Then** $P'(t) = Ce^{kt}k = kP(t)$ **as desired.**

$P(0) = C$, **so** $P(t) = P_0e^{kt}$ **is a formula for such a population.**

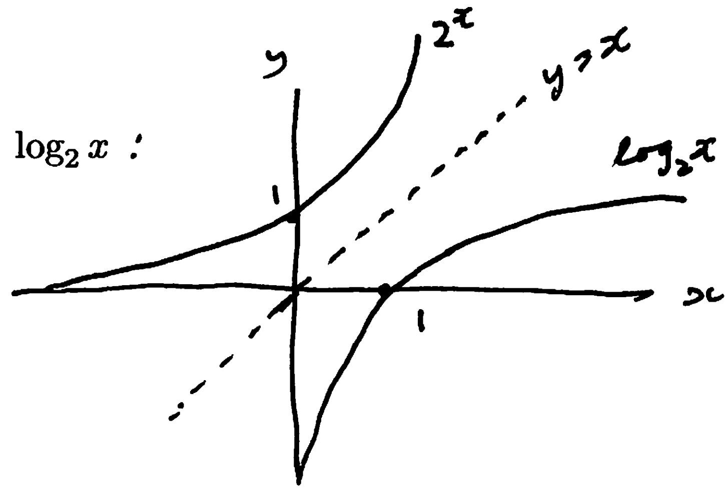
Radioactive decay: $k < 0$.

4.5 Logarithms.

If $a > 0$, $a \neq 1$, $x > 0$, **then** $y = \log_a x$ **means** $a^y = x$.

So $\log_3 9 = 2$, $\log_2(\frac{1}{8}) = -3$, $\log_a 1 = 0$ **for any base** a.

compare the graphs of 2^x **and** $\log_2 x$ **:**



properties: $a^{x+y} = a^x a^y$ **implies** $\log_a xy = \log_a x + \log_a y$

Similarly $\log_a x/y = \log_a x - \log_a y$, $\log_a(x^r) = r \log_a x$

$\log_a(a^r) = r$ **and** $a^{\log_a r} = r$ (a^r **and** $\log_a r$ **are inverse functions).**

The natural logarithm $\ln x = \log_e x$.

$$\log_a x = \frac{\ln x}{\ln a}$$

Recall that $\ln(e^x) = x$ **and** $e^{\ln x} = x$.

So $\dfrac{d}{dx} e^{\ln x} = \dfrac{d}{dx} x = 1$.

But $\dfrac{d}{dx} e^{\ln x} = e^{\ln x} \dfrac{d}{dx} \ln x = x \dfrac{d}{dx} \ln x$.

Thus $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$.

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}.$$

More generally, $\dfrac{d}{dx} \ln(f(x)) = \dfrac{f'(x)}{f(x)}$.

$$\frac{d}{dx} x \ln x = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

$$\frac{d}{dx} \frac{\ln x}{e^x} = \frac{e^x \frac{1}{x} - \ln x e^x}{(e^x)^2} = \frac{1}{xe^x} - \frac{\ln x}{e^x}.$$