11.4 Linear Systems of Differential Equations

2. $\frac{dx}{dt} = x_1 + 6x_2$

$\frac{dx_2}{dt} = 5x_1 + 2x_2$

The system can be represented in matrix form as

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find the eigenvalues.

$$\begin{aligned} 0 &= \det(M - \lambda I) \\ &= \det \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix} \\ &= (1-\lambda)(2-\lambda) - 6(5) \\ &= \lambda^2 - 3\lambda - 28 \\ &= (\lambda - 7)(\lambda + 4) \end{aligned}$$

$\lambda = -4$ and $\lambda = 7$

For $\lambda = -4$,

$$(M - \lambda I)X + \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One solution to this system is $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$.

For $\lambda = 7$,

$$(M - \lambda I)X + \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One solution to this system is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The matrix of eigenvectors is

$$P = \begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix}.$$

$$P^{-1} = \frac{1}{11} \begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix} \text{ and } P^{-1}MP = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix}$$

Now let $X = PY$ and multiply both sides of the differential equation by P^{-1}, yielding

$$\frac{dY}{dt} = P^{-1}MPY$$

or,

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

or,

$$\frac{dy_1}{dt} = -4y_1, \text{ and } \frac{dx_2}{dt} = 7y_2$$

Integrate these to yield

$$y_1 = C_1 e^{-4t} \text{ and } y_2 = C_2 e^{7t}$$

Finally, calculate $X = PY$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{-4t} \\ C_2 e^{7t} \end{bmatrix} = \begin{bmatrix} 6C_1 e^{-4t} + C_2 e^{7t} \\ -5C_1 e^{-4t} + C_2 e^{7t} \end{bmatrix}$$

or,

$$x_1 = 6C_1 e^{-4t} + C_2 e^{7t}, x_2 = -5C_1 e^{-4t} + C_2 e^{7t}$$

10. From exercise 2, the general solution is

$$\begin{aligned} x_1 &= 6C_1 e^{-4t} + C_2 e^{7t} \\ x_2 &= -5C_1 e^{-4t} + C_2 e^{7t}. \end{aligned}$$

Setting $x_1(0) = 12$ and $x_2(0) = 1$ gives

$$\begin{aligned} 12 &= 6C_1 + C_2 \\ 1 &- -5C_1 + C_2. \end{aligned}$$

Solving this system gives

$$C_1 = 1, C_2 = 6.$$

Therefore,

$$\begin{aligned} x_1 &= 6e^{-4t} + 6e^{7t} \\ x_2 &= -5e^{-4t} + 6e^{7t}. \end{aligned}$$