MAP-GERMS DETERMINED BY THEIR DISCRIMINANTS

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§0. Introduction.

A simple, geometric, necessary condition for $\mathcal{R}$-equivalence of map-germs (namely, equality of discriminants) and for $\mathcal{A}$-equivalence (equivalence of discriminants) is, for a rather large collection of map-germs, also sufficient, or at least sufficient up to a well-understood finite ambiguity. For this result to be useful, we need a detailed knowledge of which map-germs it applies to, and we need to understand the discriminant and its properties. We address both these needs here.

We will be concerned with C^τ map-germs $f : (N, S) \longrightarrow (P, y_0)$, with S finite, where N and P are C^τ manifolds of dimension n and p respectively. Here C^τ means either complex analytic ($\tau = \mathbb{C}$-ω), real analytic ($\tau = \mathbb{R}$-ω), or C^∞ ($\tau = \infty$).

(0.1) Definition. C^τ map-germs $f_i : (N_i, S_i) \longrightarrow (P_i, y_i)$, $i = 0, 1$, are $\mathcal{A}$-*equivalent* if there exists C^τ diffeomorphism-germs $r : (N_0, S_0) \longrightarrow N_1$, with $r(S_0) = S_1$, and $l : (P_0, y_0) \longrightarrow (P_1, y_1)$ such that $l \circ f_0 = f_1 \circ r$. They are called $\mathcal{R}$-*equivalent* if l is the identity.

(0.2) Definition. The *critical set* of a C^τ map $f : N \longrightarrow P$ is

$$C = C(f) = \{x \mid \operatorname{rk} d_x f < p\},$$

where $d_x f$ is the derivative of f at x. The *discriminant* of f is

$$D = D(f) = f(C(f)).$$

Critical sets and discriminants are preserved by equivalences; in the situation of (0.1),

$$r(C(f_0)) = C(f_1))$$

and

$$l(D(f_0)) = D(f_1)).$$

In particular, equality of discriminants is a necessary condition for $\mathcal{R}$-equivalence; this is the "simple geometric condition" we work with.

To describe the class of map-germs for which this necessary condition for $\mathcal{R}$-equivalence is sufficient, we need to introduce some notation and terminology.

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(0.3) Notation and terminology. $\mathcal{E} = \mathcal{E}_N$ will denote the sheaf of C^τ, E-valued function-germs on N, where $E = \mathbb{R}$ or $\mathbb{C}$ in agreement with τ. $\mathcal{E}_a$ is the stalk of $\mathcal{E}$ at $a \in N$. If $N = E^n$, we denote $\mathcal{E}_0$ by E_n and the unique maximal ideal in E_n by m_n. Let $J(f)$ be the ideal in E_n generated by the $p \times p$ minors of the Jacobian matrix of f, and let $\mathcal{J}(f)$ be the corresponding sheaf of ideals. For any set or set-germ X, $I(X)$ denotes the ideal of germs which vanish on X; $\mathcal{I}(X)$ is the sheaf of ideals of function-germs which vanish on X. If an analytic variety-germ V has irreducible components $V_1, \ldots, V_r$, then a subvariety W is said to have codimension $\geq c$ if $W \cap V_i$ has codimension $\geq c$ for each i. A property of an analytic map-germ $f : V \longrightarrow W$ is said to hold generically if it holds off a codimension ≥ 1 subvariety of V. If f is a C^∞ map-germ restricted to a set C, we say a property of f holds generically if it holds on an open dense subset of C.

(0.4) Definition. Suppose $f : (N, S) \longrightarrow (P, y_0)$ is a C^τ map-germ with S finite. f is a *critical normalization* (CN, for short) if $\mathcal{J}(f)_S = \mathcal{I}(C)_S$ and $f \mid C : C \longrightarrow D$ is a C^τ *normalization*, which means:

($\tau = \mathbb{C}$-ω) C_a is a normal variety for each $a \in S$ (see §1) and $f|C$ is finite-to-one and generically one-to-one.

($\tau = \mathbb{R}$-ω) $(f|C)_{\mathbb{C}} : C_{\mathbb{C}} \longrightarrow D_{\mathbb{C}}$ is a normalization and $C_{\mathbb{C}} = C(f_{\mathbb{C}})$ (where the subscript $\mathbb{C}$ means "complexification" - see for example [N]).

($\tau = \infty$) $f|C$ is C^∞ equivalent to a $\mathbb{R} - \omega$ normalization and, for C' any representative of C, $\dim C'_x = \min\{\dim P - 1, \dim N\}$ at each x sufficiently near S.

For more on what a normalization is, see §1. A crucial property of normalizations is their essential uniqueness. In [GW2], the following statement of this uniqueness is proven:

(0.5) Theorem. *Let $f_i : (N_i, S_i) \longrightarrow (P_i, y_i)$, $i = 0, 1$, be critical normalizations with $D(f_0) = D(f_1)$. Then there exists a C^τ-diffeomorphism germ $r : (N_0, S_0) \longrightarrow (N_1, S_1)$, unique on $C(f_0)$, such that*

$$r(C(f_0)) = C(f_1)$$

and

$$f_1 \circ r|C(f_0) = f_0|C(f_0).$$

In fact, for critical normalizations, equality of discriminants is almost enough to guarantee right-equivalence, not merely right-equivalence on the critical sets. However, in the real case we need to make one additional assumption, which we now explain.

The *quadratic differential* (or *Hessian*) $d_x^2 f$ of f is a quadratic form from the kernel to the cokernel of $d_x f$ (see Chapter 10 of [Br]). Without loss of generality, we may assume $(N, x) = (\mathbb{R}^n, 0)$ and $(P, y) = (\mathbb{R}^p, 0)$. Suppose f has rank $p - 1$ at 0. We choose an orientation of $\operatorname{cok} d_0 f$, which is one-dimensional. Then $d_0^2 f$ has a well-defined index (the dimension of the space spanned by those eigenvectors corresponding to negative eigenvalues)—that is, if r, l are C^τ-diffeomorphism-germs of $(\mathbb{R}^n, 0), (\mathbb{R}^p, 0)$, respectively, then

$$d^2(l \circ f \circ r) = \overline{d_0 l} d^2 f(d_0 r|, d_0 r|),$$

where $\overline{d_0 l} : \operatorname{cok} d_0 f \longrightarrow \operatorname{cok} d_0(l \circ f \circ r)$ is the isomorphism induced by $d_0 l$, and $d_0 r : \ker d_0 f \longrightarrow \ker d_0(l \circ f \circ r)$ is the isomorphism induced by restricting $d_0 r$.

In the particular case where $n \geq p$ and $\operatorname{corank} d_0 f = 1$, and if $\alpha : \operatorname{cok} d_0 f \longrightarrow \mathbb{R}$ is a linear isomorphism, then $\alpha \circ d_0^2 f$ is a quadratic form, so has a well-defined index. Indeed, we see that this index depends only on a choice of orientation (i.e. positive direction) in $\operatorname{cok} d_0 f$, and is independent of any choice of coordinates in $\mathbb{R}^n$. Thus it is, once a choice of orientation has been made, an $\mathcal{R}$-invariant of f.

It is very easy to calculate in local coordinates; a map-germ $f : (N, x_0) \longrightarrow (P, y_0)$ with $\dim \operatorname{cok} d_{x_0} f = 1$ is $\mathcal{A}$-equivalent to a map-germ $g : (E^n, 0) \longrightarrow (E^p, 0)$ of the form

$$g(u_1, \ldots, u_k, y_1, \ldots, y_b, z_1, \ldots, z_c) = (u_1, \ldots, u_k, \sum_{j=1}^{b} Q(y) + F(u, z)),$$

where Q is a non-degenerate quadratic form, and $F \in m_z E_{u,z}$; the index of the Hessian of f is, with the appropriate choice of orientation, the index of Q.

It should perhaps be remarked that the index of the Hessian is not an $\mathcal{A}$-invariant: for example, the map-germs from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}^2, 0)$ given by

$$(u, y, z) \mapsto (u, y^2 + z^3 + uz),$$

and

$$(u, y, z) \mapsto (u, -y^2 + z^3 + uz)$$

are $\mathcal{A}$-equivalent but not $\mathcal{R}$-equivalent. The signature *is* $\mathcal{A}$-invariant, but the above example shows also that equal signature and equal discriminant *does not* imply $\mathcal{R}$-equivalence.

(0.6) Theorem. *(*[duPW1]*). Let $f_i : (N_i, S_i) \longrightarrow (P_i, y_i)$, $i = 0, 1$, be critical normalizations with $D(f_0) = D(f_1)$. Then*

1. ($\mathbb{C}$-ω) *f_0 and f_1 are right-equivalent;*
2. ($\mathbb{R}$-ω *or* ∞)
   a) if $n < p$ or corank $d_x f_0 \neq 1$ for all $x \in S_0$, then f_0 and f_1 are right-equivalent
   b) if $n \leq p$ and corank $d_x f_0 = 1$ for some $x \in S_0$, then there exists a bijection $\sigma : S_0 \longrightarrow S_1$ such that $\sigma(S_0') = (S_1')$ (where $S_i' = \{x \in S_i :$ corank $d_x f_i = 1\}$) and such that, for all $x \in S_0'$:
      i) $\operatorname{Im} d_x f_0 = \operatorname{Im} d_{\sigma(x)} f_1$ *, and*
      ii) $\operatorname{rk} d_x^2 f_0 = \operatorname{rk} d_{\sigma(x)}^2 f_1$.

   Give $\operatorname{cok} d_x f_0$ and $\operatorname{cok} d_{\sigma(x)} f_1$ the same orientation for all $x \in S_0'$. Suppose that $d_x^2 f_0$ and $d_{\sigma(x)}^2 f_1$ have the same index for all $x \in S_0'$. Then f_0 and f_1 are right-equivalent.

(0.7) Remark. If we replace the condition $D(f_0) = D(f_1)$ by $l(D(f_0)) = D(f_1)$ for some diffeomorphism-germ l, then we easily get from the above theorem necessary and sufficient conditions for two critical normalizations to be equivalent.

One reason for emphasizing right-equivalence of germs, as we do, is that this more easily leads to global equivalence theorems. This is because the local right equivalences are relatively few in number, and hence it is more easy to see when they can be pieced together. For example, when $n < p$, the local right equivalences are unique, and hence automatically piece together globally (for proper maps which are locally critical normalizations). Global equivalence theorems have been derived using this approach in [GW2]. The calculation of the set of local right equivalences between two critical normalizations has been made in [duPW2].

The question immediately posed by (0.6) is: when is a map-germ a CN? We give an answer in terms of "weak transversality conditions" which is very useful for calculations and for relating the notion of a CN to other families of map-germs studied in singularity theory. We will prove:

(0.8) Theorem. *Let $f : (N, S) \longrightarrow (P, y_0)$ be a C^τ map-germ.*

1) *($\tau = \mathbb{C}$-ω) f is a critical normalization if, and only if*
   a) *$f \pitchfork \{0\}$ on $(N - S, S)$;*
   b) *$j^1 f \pitchfork$ all Thom-Boardman singularities off a codimension 2 subset of C;*
   c) *${}_2j^1 f \pitchfork$ all multijet Thom-Boardman singularities off a codimension 1 subset of C.*
2) *($\tau = \mathbb{R}$-ω) f is a critical normalization if, and only if*
   a) *the complexification $f_{\mathbb{C}}$ of f is a $\mathbb{C}$-ω CN;*
   b) *$C(f_{\mathbb{C}}) = C(f)_{\mathbb{C}}$.*
3) *($\tau = C^\infty$) f is a critical normalization if its C has dimension $\min\{p-1, n\}$ at each point and it is $\mathcal{A}$- equivalent to an analytic map-germ whose complexification is a CN.*

The result in (0.8.1) is quite satisfactory; it shows that critical normality is a condition holding in general to complex-analytic map-germs whenever $\dim P > 2$, (and also when $\dim N = 1$ and $\dim P = 2$).

The results of (0.8.2) and (0.8.3) show that being a CN is not so general in the real case; but it still follows from them that many of the germs commonly considered in singularity theory are critical normalizations and so are determined by their discriminants: this includes all map-germs multi-transverse to the first order Thom-Boardman varieties Σ^i which are $\mathcal{A}$-equivalent to analytic germs, and so in particular all C^0-stable germs $\mathcal{A}$-equivalent to analytic germs (by results of [duPWa]), and all C^∞-stable germs (a result previously obtained by Wirthmüller in [Wir]). The Lagrangean stable map-germs (see [Ar1]) are also CNs. Also, all $\mathcal{A}$-finite map-germs $f : (N, S) \longrightarrow (P, y_0)$ with no point of S isolated in $C(f)$ and $\dim P > 2$ are CN; indeed, all such ∞-C^0-$\mathcal{A}$-determined map-germs $\mathcal{A}$-equivalent to an analytic germ are CNs (by results of du Plessis "in preparation").

While Theorem (0.6) is fairly general, it is still not as general as we would like. For example, if $p = 1$, critical normalizations are just Morse functions and (0.6) is just the Morse Lemma. Golubitsky and Guillemin [GG] show that $\mathcal{A}$-finite function-germs f, g are $\mathcal{R}$-equivalent iff there is an algebra isomorphism $E_n/J(f)^2 \cong E_n/J(g)^2$ sending the projection of f to that of g; a simpler proof is given in [duPW1].

When $p = 2$, those finitely $\mathcal{A}$-determined germs f which have rank 1 and are transverse to Σ^{n-1} are critical normalizations, whereas all other finitely $\mathcal{A}$-determined germs in these dimensions are not. Indeed, such map-germs are *not* determined by their discriminants, even in the $\mathcal{A}$-finite $\mathbb{C}$-ω case, as the following examples show.

(0.9) Examples.
1). The map-germs $(E^3, 0) \longrightarrow (E^2, 0)$ given by

$$(u, x, y) \mapsto (u, x^3 + y^3 + 3u^5(x - y)),$$
$$(u, x, y) \mapsto (u, x^2 + 4y^5 + 5u^6 y)$$

both have discriminant $t^4 + 64u^{30} = 0$.

For an example whose discriminant in the real case is not a point, we need to work somewhat harder:
2). The map-germ

$$(u, x, y) \mapsto (u, x^3 + y^3 + 3u^5(ax + by))$$

has discriminant

$$t^4 + 8(a^3 + b^3)u^{15}t^2 + 16(a^3 - b^3)^2 u^{30} = 0,$$

while

$$(u, x, y) \mapsto (u, x^2 + y^5 + cu^6 y + du^3 y^3)$$

has discriminant

$$3 \cdot 125t^4 + (108d^5 - 900cd^3 + 2000c^2 d)u^{15}t^2 + (256c^5 - 128c^4 d^2 + 16c^3 d^4)u^{30} = 0,$$

which are equal if

$$3 \cdot 125 \cdot 8(a^3 + b^3) = 108d^5 - 900cd^3 + 2000c^2 d, \text{ and}$$
$$3 \cdot 125 \cdot 16(a^3 - b^3)^2 = 256c^5 - 128c^4 d^2 + 16c^3 d^4.$$

These two equations have many real solutions with a, b of the same sign.
3). The map-germ $(E^2, 0) \longrightarrow (E^2, 0)$ given by

$$(x, y) \mapsto (-x^2 + y^3, -y^2 + x^3)$$

and the map-germ $(E^2, 0) \sqcup (E^2, 0) \longrightarrow (E^2, 0)$ given by

$$(x, y) \mapsto (x, \frac{1}{2}y^3 + \frac{3}{2}xy)$$

on the first component, and by

$$(x, y) \mapsto (\frac{1}{2}y^3 + \frac{3}{2}xy, x)$$

on the second, have the same discriminant, namely

$$(s^2 + t^3)(s^3 + t^2) = 0.$$

4). The map-germs $(E^2, 0) \longrightarrow (E^2, 0)$ given by

$$(x, y) \mapsto (x, -\frac{1}{6}y^6 + xy),$$
$$(x, y) \mapsto (xy, \frac{1}{3}x^3 + \frac{1}{2}y^2)$$

both have discriminant

$$5^5 s^6 - 6^5 t^5 = 0.$$

As these examples show, in the case $n \geq p = 2$, the discriminant cannot in general predict the number of source points, or the rank, or in the cokernel rank one case, the rank of the Hessian, even for $\mathbb{C} - \omega$ $\mathcal{A}$-finite map-germs. It turns out that a further invariant is required, the so-called conductor ideal suffices ([duP3]), but carries more information than necessary ([duP4]); we refer to these papers for further information.

Although most map-germs in the case $n \geq p = 2$ are not CN's, still most map-germs are $CS - FST$ (critical simplification of finite singularity type; see (2.1) for the definition); and although a CS-FST is not determined by its discriminant, a family of CS-FST germs is (this is proved in [BduPW]). The CS-FST map-germs are characterized by an analogue of Theorem (0.8), in which condition (1b) is changed by replacing "codimension 2" by "codimension 1".

By work of Gaffney ([Ga]), if $n = p = 2$, a family of CS-FST map-germs is determined up to C^0-$\mathcal{A}$ equivalence by the C^0-equivalence type of its discriminant. This relates to an important question: since the analytic type of the discriminant determines the analytic type of the map-germ (for CN's, and for CS-FST's if we deal with families), to what extent does the topological type of the discriminant determine the topological type of the map-germ (or family of map-germs)?

Another question posed by (0.8) in the C^∞ case is to what extent the requirement of $\mathcal{A}$-equivalence to an analytic germ might be relaxed. It turns out that it can be avoided completely for cokernel rank one map-germs $f : (N, S) \longrightarrow (P, y_0)$, $\dim N > \dim P$, transverse to Σ^{n-p+1}; such map-germs with $f|C(f)$ generically one-to-one are determined by their discriminant. This is described in [duP1], together with the rather different line of development [Hö], [Te], [Ph] and [duPW1] leading to it.

The preceding discussion of possibilities of extensions to the theory should not be allowed to obscure the fact that it is already of very wide relevance. It allows us to concentrate on properties of the discriminant when studying $\mathcal{R}$- or $\mathcal{A}$-equivalence of many map-germs. It seems that both geometric and algebraic properties are relevant. A geometric use of determining discriminants, to describe maximal compact or reductive subgroups of $\mathcal{A}$-symmetries of a CS is to be found in [duPW2].

In this paper we concentrate on the algebraic properties of discriminants. In particular, we give recipes for finding the equations of the discriminant, at least in the cases where $n \geq p-1$. In the process we see that these equations appear as the determinants of maximal minors of a matrix (a "discriminant matrix") very closely related to the $\mathcal{A}_e$-tangent space, setting up a theory allowing much $\mathcal{A}$-classification. This will be exploited in [duP5].

We would like to acknowledge our indebtedness to ideas from other workers in this area. In particular, the survey article of Teissier [Te] has had a very considerable influence, but also ideas from Arnol'd [Ar2], Saito [Sa] and Wirthmüller [Wir] have been useful to us.

Since the announcement of the results of this article in [GW1], a number of other authors have written independently on various aspects of the algebraic geometry of discriminants; we cite Bruce [Bru], Mond/Pellikaan [MP], Looijenga [Lo] and Terao [Ter]. We have also profited from their work.

This paper is organized as follows. In §1, we provide some background material in commutative algebra. In §2, we exploit this to characterize critical normalizations in terms of "weak transversality conditions" (i.e. we prove Theorem (0.8)), and we define and similarly characterize some other map-germ classes. A number of the results in this section appear, usually with more restrictive hypotheses, in Chapter 5 of [Lo]. In §3, we give recipes for calculating discriminants and discriminant matrices, prove a result on the naturality of these discriminant matrices, and discuss relations to $\mathcal{A}$-classification questions. In particular, we prove a generalization of a theorem of Damon about the $\mathcal{A}$-codimension of a map-germ ([Da]).

§1 Some commutative algebra.

In this section, R will always be a commutative, Noetherian local ring, I an ideal of R, m the maximal ideal of R, and M a finitely generated R module.

(1.1) Definitions.

1) A *finite free resolution* of M of length l is an exact sequence

$$0 \longrightarrow R^{k_l} \longrightarrow \cdots \longrightarrow R^{k_1} \longrightarrow R^{k_0} \longrightarrow M \longrightarrow 0.$$

The *homological dimension of* M, denoted $\operatorname{hd}_R M$, is the minimum of the lengths of all finite free resolutions of M. If no such finite free resolution exists, $\operatorname{hd}_R M = \infty$. ($\operatorname{hd}_R M$ is equal to the analogously defined projective dimension of M; see for example the proof of Theorem 8 of Chapter IV of [Se]).

2) A sequence $a_1, \dots, a_k$ of elements of I is called an *M-sequence in I* if, for each i between 1 and k, a_i is not a zero-divisor of $M/(a_0, \dots, a_{i-1})M$, where $a_0 = 0$. The M-sequence $a_1, \dots, a_k$ in I is maximal if there does not exist any $b \in I$ such that $a_1, \dots, a_k$, b is an M-sequence. It is a fact (see (15.B) of [Mats]) that all maximal M-sequences in I have the same length; this length is called the *I-depth of* M, and is denoted $\operatorname{depth}_R(I, M)$. The *depth of* M is $\operatorname{depth}_R(m, M)$, abbreviated $\operatorname{depth}_R(M)$.

3) A chain $P_0 \subset P_1 \subset \cdots \subset P_r$ of prime ideals in R, with $P_i \neq P_{i+1}$ for all i, is said to have length r. If P is a prime ideal in R, the *height of* P, denoted $\operatorname{ht} P$, is the upper bound of the lengths of the chains of prime ideals contained in P. The height of I is the lower bound of the heights of all prime ideals containing I. The *dimension of* R, denoted $\dim R$, is the height of m. The *dimension of* M is the dimension of $R/\operatorname{ann} M$, where $\operatorname{ann} M = \{r \in R : rM = 0\}$ is the *annihilator of* M.

(1.2) Proposition. *(Proposition 12 of Chapter IV of* [Se]*).*

Suppose $f : R \longrightarrow S$ is a homomorphism of local rings making S into a finitely generated R module. If M is a finitely generated S module, then $\operatorname{depth}_R M = \operatorname{depth}_S M$ *and* $\dim_R M = \dim_S M$.

(1.3) Definition and remarks R is a *regular local ring* if m is generated by $\dim R$ elements. By Proposition 9 of Chapter IV of [Se], R is regular if, and only if, $\operatorname{hd}_R(R/m)$ is finite. In this case, $\mathrm{hd}_R(M)$ is finite for every finitely generated R-module M. Furthermore, $\mathrm{depth}_R R = \dim R = \mathrm{hd}_R(R/m)$.

(1.4) Proposition. *(Proposition 21 of Chapter IV of* [Se]*.*
If R is a regular local ring and M is a finitely generated R-module, then $\mathrm{hd}_R(M) + \mathrm{depth}_R(M) = \dim R$.

(1.5) Remarks. E_n ($\tau = \mathbb{R}$-ω or $\mathbb{C}$-ω) is a regular local ring of dimension n. Suppose I is an ideal of E_n, $R = E_n/I$, and $V = V(I)$. Let $\dim V$ be the set-theoretic dimension of V (it is the largest d such that there exist regular points of dimension d of V arbitrarily near 0); if the irreducible components of V are $V_1, \ldots, V_r$, then $\dim V$ is the largest dimension of the V_i's. When $\tau = \mathbb{C}$-ω, $\dim R = \dim V$ and $\mathrm{ht}\, I = \mathrm{cod}\, V = n - \dim V$.

(1.6) Definition. M is *Cohen-Macaulay* if $\dim M = \mathrm{depth}_R M$. R is a Cohen-Macaulay ring if it is Cohen-Macaulay as an R-module (that is, $\mathrm{ht}\, m = \mathrm{depth}_R(m, R)$).

(1.7) Proposition. *Suppose R is a regular local ring. Let M be a finitely generated R module. Then M is Cohen-Macaulay if, and only if,* $\mathrm{hd}_R M = \mathrm{ht}(\mathrm{ann}\, M)$. *In particular, if $\tau = \mathbb{C}$-ω and $R = E_n$, then M is Cohen-Macaulay if, and only if,* $\mathrm{hd}_R M = \mathrm{cod}\, V(\mathrm{ann}\, M)$.

Proof. M is Cohen-Macaulay as an R-module if, and only if, $\mathrm{depth}_R M = \dim M = \dim R/\mathrm{ann}\, M = \dim R - \mathrm{ht}(\mathrm{ann}\, M)$ (by I.6.12 of [To]). By (1.4), $\mathrm{hd}_R(M) = \dim R - \mathrm{depth}_R M$. □

(1.8) Corollary. *Suppose R is a regular local ring and M is a finitely generated R module with* $\dim M = \dim R$. *Then M is Cohen-Macaulay if, and only if, M is a free R-module. In particular, R itself is Cohen-Macaulay.*

(1.9) Proposition. *(Theorem 136 of* [K]*).*
If R is Cohen-Macaulay, then for every ideal I, $\mathrm{ht}\, I = \mathrm{depth}_R(I, R)$.

(1.10) Corollary. *If $R = E_n$, $\tau = \mathbb{C}$-ω, then* $\mathrm{depth}_R(I, R) = \mathrm{cod}\, V(I)$.

(1.11) Definition. For each $a \in M$, let $\mathrm{ann}(a) = \{r \in R : ra = 0\}$. Those ideals $\mathrm{ann}(a)$ which are prime are called *associated prime ideals of M*. The set of these is denoted $\mathrm{ass}\, M$. (The union of the associated primes of M is the set of zero divisors of M, and the intersection of the associated primes is $\mathrm{ann}\, M$). M is *unmixed* if $\dim R/P = \dim M$ for all $P \in \mathrm{ass}\, M$.

(1.12) Proposition. *(Proposition 13 of Chapter IV of* [Se]*). If M is Cohen-Macaulay, then M is unmixed.*

Suppose $\varphi : R^n \longrightarrow R^p$ is an R-module morphism; φ can be interpreted as a matrix. Let $I(\varphi)$ be the ideal in R generated by the $p \times p$ minors of φ.

(1.13) Proposition. *(Corollaries 2.5 and 2.7 of* [BR]*).*

1) $\mathrm{depth}_R(I(\varphi), R) \leq n - p + 1$
2) *If* $\mathrm{depth}_R(I(\varphi), R) = n - p + 1$, *then* hd_R *cok* $\Lambda^k \varphi = n - p + 1$ *for all k, $1 \leq k \leq p$.*

The cases of most interest to us are $\operatorname{cok}\Lambda^1\varphi = R^p/\varphi(R^n)$ and $\operatorname{cok}\Lambda^p\varphi = R/I(\varphi)$.

Since R is Noetherian, M finitely generated implies M is finitely presented, i.e., there exists an exact sequence

$$R^n \xrightarrow{\varphi} R^p \longrightarrow M \longrightarrow 0.$$

The 0^{th} *Fitting ideal* of M is defined to be $F_0(M) = I(\varphi)$. This ideal does not depend on the presentation chosen.

(1.14) Proposition. *Suppose R is a regular local ring, M is a finitely generated R-module,* $\operatorname{depth}_R M = \dim R - 1$ *and* $\operatorname{ann} M \neq 0$. *Then $F_0(M)$ is generated by one element, which is not zero.*

Proof. (from [Te], bottom of page 614). By (1.4), $\operatorname{hd}_R M = 1$, so there is a resolution

$$0 \longrightarrow R^s \xrightarrow{\psi} R^r \longrightarrow M \longrightarrow 0.$$

Thus $s \leq r$. Since $\operatorname{ann} M \neq 0$, $M \otimes_R K = 0$, where K is the field of fractions of R. Thus $\psi \otimes 1_K : K^s \longrightarrow K^r$ is surjective, so $s \geq r$. Thus $s = r$ and $F_0(M) = (\det\psi)R$. $\square$

Let $\mathcal{M}$ be a coherent sheaf of $\mathcal{E}$ modules, $\tau = \mathbb{R}$-ω or $\mathbb{C}$-ω. Let $\mathcal{P}_{a,i}$'s be the associated primes of $\mathcal{M}_a$ and let $V_{a,i} = V(\mathcal{P}_{a,i})$. The $V_{a,i}$'s are not necessarily the same as the irreducible components of $(\operatorname{supp}\mathcal{M})_a$, even in the case $E = \mathbb{C}$ and $\mathcal{M} = \mathcal{E}/\mathcal{I}$. For example, let $n = 2$ and let $\mathcal{I}$ be the sheaf generated by x^2 and xy in $\mathcal{E}$. The associated primes are $x\mathcal{E}$ and $(x, y)\mathcal{E}$, so $V_{0,1} = \{y - \text{axis}\}$ and $V_{0,2} = \{0\}$. Such embedded primes do not occur if $\mathcal{M}_a$ is unmixed.

The proof of the next proposition is a simplification of the proof of Malgrange's Theorem on $\mathcal{M}$-dense sets (see VI.2.4 of [To]), which is the analogous result for $\tau = \infty$.

(1.15) Proposition. *Suppose ξ is a section in $\mathcal{M}$ near a. Then $\xi(a) = 0$ if, and only if, for each i and any representative $\widetilde{V}_{a,i}$ of $V_{a,i}$, there exists in $\widetilde{V}_{a,i}$ a sequence $x_j \longrightarrow a$ such that $\xi(x_j) = 0$ for all j.*

Proof. "Only if" is trivial, since $\xi(a) = 0 \Rightarrow \xi(x) = 0$ for all x near a, by coherence of $\mathcal{M}$. To prove "if", we assume $\xi = 0$ on X, the germ at a of a set satisfying: for all i, $X \cap V_{a,i} \neq \emptyset$.

By the reduced primary decomposition of 0 in $\mathcal{M}_a$ (see (I.3.2) of [To]), there exist submodules $\mathcal{M}_{a,i}$ of $\mathcal{M}_a$ such that $\cap\mathcal{M}_{a,i} = \{0\}$ and the $\mathcal{M}_a/\mathcal{M}_{a,i}$ are $\mathcal{P}_{a,i}$-coprimary (this means that, for each $r \in \mathcal{P}_{a,i}$, there is an s such that $r^s \cdot \mathcal{M}_a \subset \mathcal{M}_{a,i}$ — i.e., $\mathcal{P}_{a,i} = \sqrt{\operatorname{ann}(\mathcal{M}_a/\mathcal{M}_{a,i})}$, and for each $r \notin \mathcal{P}_{a,i}$, $r\cdot : \mathcal{M}_a/\mathcal{M}_{a,i} \longrightarrow \mathcal{M}_a/\mathcal{M}_{a,i}$ as injective.) If we can show $\xi(a) \in \mathcal{M}_{a,i}$ for all i, then $\xi(a) = 0$ and we are done.

Thus we are reduced to proving the Proposition in the case $\mathcal{M}_a$ is $\mathcal{P}_a$-coprimary. Since $\mathcal{P}_a$ is the only associated prime of $\mathcal{M}_a$, our assumption is the $\xi = 0$ on X, the germ of a set at a such that $X \cap V_a \neq \emptyset$, where $V_a = V(\mathcal{P}_a)$.

There exists a composition series (see (I.3.8) of [To]): $0 = \mathcal{N}_{a,0} \subset \mathcal{N}_{a,1} \subset \cdots \subset \mathcal{N}_{a,p+1} = \mathcal{M}_a$ such that $\mathcal{N}_{a,i+1}/\mathcal{N}_{a,i} \cong \mathcal{E}_a/\mathcal{Q}_{a,i}$, where $\mathcal{Q}_{a,i}$ is a prime ideal containing P_a.

Let $\mathcal{N}_i$ (resp. $\mathcal{G}_i$, $\mathcal{P}$) be coherent subsheaves of $\mathcal{M}$ (resp. $\mathcal{E}$) satisfying $(\mathcal{N}_i)_a = \mathcal{N}_{a,i}$, $(\mathcal{Q}_i)_a = \mathcal{Q}_{a,i}$, $(\mathcal{P})_a = \mathcal{P}_a$. By coherence, $\mathcal{N}_{i+1}/\mathcal{N}_i \simeq \mathcal{E}/\mathcal{Q}_i$ and $\mathcal{Q}_i \supset \mathcal{P}$ near a.

Suppose we have proven that there is a $g_{i+1} \in \mathcal{E}$ such that $(g_{i+1})_a \notin \mathcal{P}_a$ and $(g_{i+1}\xi)_a \in \mathcal{N}_{a,i+1}$. If $\mathcal{Q}_{a,i} \neq \mathcal{P}_a$, then there is an $h \in \mathcal{E}$ with $h_a \in \mathcal{Q}_{a,i} \setminus \mathcal{P}_a$. Since there is an isomorphism $\Phi : \mathcal{N}_{i+1}/\mathcal{N}_i \longrightarrow \mathcal{E}/\mathcal{Q}_i$, Φ^{-1} given by multiplication, $(hg_{i+1}\xi)_a \in \mathcal{N}_{a,i}$. Let $g_i = hg_{i+1}$.

Now suppose $\mathcal{Q}_{a,i} = \mathcal{P}_a$. Since $\mathcal{P}_a$ is prime, $\dim(\mathcal{E}_x/\mathcal{P}_x)$ is a constant d for x near a (see (II.7.2) of [To]). Let $\mathcal{K}$ be the coherent sheaf generated by $\mathcal{P}$ and ζ, where $\overline{\zeta} = \Phi(\overline{g_{i+1}\xi})$. For $x \in X$, $\xi_x = 0$, so $\zeta_x \in (\mathcal{Q}_i)_x = \mathcal{P}_x$, so $\dim(\mathcal{E}_x/\mathcal{K}_x) = d$. By a Theorem of Tougeron (see (II.5.3) of [To]), $x \longmapsto \dim(\mathcal{E}_x/\mathcal{K}_x)$ is upper-semi continuous. Thus $\dim(\mathcal{E}_a/\mathcal{K}_a) \geq d$. On the other hand, $\mathcal{P}_a$ prime and $\mathcal{K}_a \supset \mathcal{P}_a$ implies $\dim(\mathcal{E}_a/\mathcal{K}_a) \leq d$ with "=" if and only if, $\mathcal{K}_a = \mathcal{P}_a$. Thus $\zeta_a \in \mathcal{P}_a$, i.e., $(g_{i+1}\zeta)_a \in \mathcal{N}_{a,i}$. Let $g_i = g_{i+1}$.

Thus there is a $g = g_0$ with $g_a \notin \mathcal{P}_a$ and $(g\zeta)_a = 0$. But multiplying by g_a induces an injective map from $\mathcal{M}_a$ to $\mathcal{M}_a$ (since $\mathcal{M}_a$ is $\mathcal{P}_a$-coprimary). Thus $\zeta_a = 0$. □

(1.16) Definition. $\mathcal{E}_a/\mathcal{I}_a$ is said to be *reduced* (if $\tau = \mathbb{C}$-ω) or *real-reduced* (if $\tau = \mathbb{R}$-ω) if $\mathcal{I}_a = I(V(\mathcal{I}_a))$.

Applying (1.15) to $\mathcal{M} = \mathcal{E}/\mathcal{I}$, we immediately have:

(1.17) Corollary. *(Criterion for being reduced or real-reduced). $\mathcal{I}_a = I(V(\mathcal{I}_a))$ if, and only if, for each i there exists in $\widetilde{V}_{a,i}$ a sequence $x_j \longrightarrow a$ such that $\mathcal{I}_{x_j} = I(V(\mathcal{I}_{x_j}))$ for all j.*

(1.18) Corollary. *($\tau = \mathbb{C}$-ω)*

Suppose $\mathcal{E}_a/\mathcal{I}_a$ is Cohen-Macaulay. Then $\mathcal{E}_a/\mathcal{I}_a$ is reduced if, and only if, $\mathcal{E}_x/\mathcal{I}_x$ is reduced at all regular points of $V(\mathcal{I}_a)$ if and only if, $\mathcal{E}_x/\mathcal{I}_x$ is reduced at all x in a Zariski dense set of regular points of $V(\mathcal{I}_a)$.

(1.19) Definition. R is *normal* if it is an integral domain and is integrally closed in its field of fractions. An analytic variety V is normal if $E_n/I(V)$ is.

(1.20) Proposition. *(See II.3.7, II.7.9 and II.7.11 of* [To]*).*

Suppose V is a complex analytic variety and is equidimensional (all its irreducible components have the same dimension). Then V is normal if, and only if, both of the following hold:

1) *the singular set of V has codimension ≥ 2 in V*
2) $\operatorname{cod}\{x | \operatorname{hd}_{\mathcal{E}_x}(\mathcal{E}_x/I(V_x)) > i + \operatorname{cod} V\} \geq i + 3$, $\forall i \geq 3$

(where, by convention, $\operatorname{cod}\emptyset = \infty$).

Let $R_V = E_n/I(V)$. By (1.7), R_V is Cohen-Macaulay if, and only if, $\operatorname{cod} V = \operatorname{hd}_{E_n} R_V$.

(1.21) Corollary. *($\tau = \mathbb{C}$-ω) Suppose R_V is Cohen-Macaulay. Then V is normal if, and only if, its singular set has codimension ≥ 2 in V.*

§2 Critical behavior and weak transversality conditions.

In this section, we will define several types of critical behavior of map-germs, and then demonstrate how these types of critical behavior are related to transversality conditions.

(2.1) Definitions. Let $f : (E^n, 0) \longrightarrow (E^p, 0)$ be a C^τ map-germ, with $\tau = \infty$, $\mathbb{R}$-ω or $\mathbb{C}$-ω.

1) For $n \geq p$, f *has non-degenerate critical set* if $J(f) = I(C)$ and $\dim C = p - 1$; for $n < p$, we will consider all map-germs to have non-degenerate critical set.
2) f is *generically finite-to-one on its critical set* if it has non-degenerate critical set and $f|C$ has discrete fibers generically (off of a codimension 1 subvariety of C, the fibers of f are discrete sets).
3) f is a *critical simplification* (CS) if it has non-degenerate critical set and is generically one-to-one on its critical set.
4) f is of *finite singularity type* (FST) if $E_n/J(f)$ is finitely generated as an f^*E_p module.

By the Malgrange Preparation Theorem, f is FST if, and only if, $E_n/(J(f)+f^*m_pE_n)$ is a finite dimensional E vector space; in fact, by Nakayama's Lemma, $\alpha_1, \dots, \alpha_k$ generate $E_n/J(f)$ as an f^*E_p module if, and only if, their projections generate $E_n/(J(f)+f^*m_pE_n)$ as an E vector space. A complex analytic f is of FST if, and only if, $f|C(f)$ is finite-to-one if, and only if, $f^{-1}(0) \cap C(f) = \{0\}$ if, and only if, $f \pitchfork \{0\}$ on the germ of $E^n - \{0\}$ at $\{0\}$. Thus being of FST is a "weak transversality condition". If $n \geq p$ and f is FST, then $\dim C(f) = \dim D(f) = p - 1$.

The concept of a critical normalization CN was defined in (0.4). Certainly, $CN \Rightarrow CS \Rightarrow$ generically finite-to-one on its critical set $\Rightarrow$ non-degenerate critical set. Also, $CN \Rightarrow FST$.

In case $p = 1$, f has non-degenerate critical set if, and only if, $J(f) = m_n$, i.e., f has a non-degenerate critical point at 0. Then f is trivially a CN as well. Theorem 0.6 in this case is simply the Morse Lemma. If $n < p$, then C is the entire domain, and critical normalizations are the same as normalizations. Theorem (0.6) is in this case the Uniqueness of Normalization Theorem, well-known in the analytic case, proved in [GW2] in the C^∞ case (we will point out at the end of this section that the present definition of normalization implies that of [GW2]).

We first present several examples to clarify the concepts in (2.1). These examples are valid for $\tau = \infty$, $\mathbb{R}$-ω or $\mathbb{C}$-ω. We will not prove the assertions made in these examples; the proofs are easy once one has the results on transversality conditions proved later in this section.

(2.2) Examples (in each of the following, we intend f to be the germ at 0 of the indicated mapping).

1) $f(u, z) = (u, z^3)$ is FST but has degenerate critical set.
2) $f(v, x) = (v, vx)$ has non-degenerate critical set but is not generically finite-to-one on its critical set.
3) $f(x, y) = (x^2, y^2)$ is generically finite-to-one on its critical set and FST, but is not a CS.
4) $f(v_1, v_2, x_1, x_2) = (v_1, v_2, v_1x_1 + v_2x_2 + x_1^m x_2^n)$, $m \geq 1$, $n \geq 2$, is a CS but not FST (in fact, $D(f)$ is not even the germ of a closed set).
5) Any finitely A-determined germ from E^n to E^2 which is either of rank 0 at 0, or is of rank 1 but is not transverse to Σ^{n-1} at 0, is a CS but not a CN. Two such map-germs are $f(u, z) = (u, u^2z - z^3)$ and $g(x, y) = (x^2 + y^3, y^2 + x^3)$.

The main results of [duPW2] applied to map-germs which were CS's but not necessarily FST. However, in the present paper we will only deal with CS's which are also FST, and we will refer to these as CS-FST map-germs.

Next we recall some results from [duPW1] which we will need. Lemmas (2.3) and (2.4) are part of Lemma (1.2) of [duPW1].

Let E_n^k denote the k-tuples of elements of E_n (which we sometimes identify with C^τ map-germs from $(E^n, 0)$ to E^k).

(2.3) Lemma. *For any $f \in E_n^p$, $J(f) \subseteq \operatorname{ann}(E_n^p/df E_n^n) \subseteq I(C(f))$.*

(2.4) Lemma. *If $f \in E_n^p$ $(n \geq p)$ has rank $d_0 f < p-1$, then $J(f) \subset m_n^2$ and $J(f)E_n^p \subset df(m_n E_n^n)$.*

The following is (3.1) of [duPW1]:

(2.5) Lemma. *Suppose $f \in E_n^p$ $(n \geq p)$ has $\operatorname{rk} d_0 f = p-1$. Then C^τ coordinates can be chosen which put f in the form*

$$f(u, v, x, y, z) = (u, v, f'(u, v, x, y, z))$$

where $(u, v, x, y, z) = (u_1, \ldots, u_q, v_1, \ldots, v_r, x_1, \ldots, x_r, y_1, \ldots, y_s, z_1, \ldots, z_t)$, with

$$f'(u, v, x, y, z) = \sum_{i=1}^{r} v_i x_i + \sum_{j=1}^{s} \pm\, y_j^2 + F(u, x, z)$$

and with $F \in m_{(x,z)} m_{(u,x,z)}^2$ (we will only use that $F \in m_n^3$ and doesn't depend on y).

Note that $d_0^2 f$ can be identified with $\sum\limits_{j=1}^{s} \pm\, y_j^2$.

We denote the Thom-Boardman singularities by Σ^I, I a multiindex (see for example [Math1]). In particular, $j^1 f(x) \in \Sigma^i$ means that $\dim\ker d_x f = i$. We write $j^2 f(x) \in \Sigma_t^i$ if $j^1 f(x) \in \Sigma^i$ and $j^1 f \pitchfork_x \Sigma^i$. The following is part of (3.2) of [duPW1].

(2.6) Lemma. *Suppose $f \in E_n^p$, $n \geq p$. The following are equivalent:*

1) *$j^2 f(0) \in \Sigma_t^{n-p+1}$;*
2) *C is a manifold-germ of dimension $p-1$ and $I(C) = J(f)$;*
3) *f can be expressed in C^τ local coordinates as:*

$$f(u, v, x, y) = (u, v, \sum_{i=1}^{r} v_i x_i + \sum_{j=1}^{s} \pm\, y_j^2 + F(u, x)), \quad F \in m_n^3.$$

The next result is (5.2) of [duPW1].

(2.7) Lemma. *If C is a nonempty manifold-germ and $J(f) = I(C)$, then it is not possible to have $\dim C < p-1$.*

It is however possible to have $\dim C > p-1$.

(2.8) Example $f(u, y, z) = (u, y^2)$ is a map-germ with C a manifold-germ of dimension 2 and $I(C) = J(f)$. Of course, any map-germ with rank $df < p$ everywhere satisfies $I(C) = J(f) = \{0\}$.

(2.9) Lemma. *Supose $f \in E_n^p$, $n \geq p$. The following are equivalent:*

1) $\dim_E E_n/(J(f) + f^* m_p E_n) = 1$;
1′) $\dim_E E_n^p/(df(E_n^p) + f^* m_p E_n^p) = 1$;
2) $j^2 f(0) \in \Sigma_t^{n-p+1}$ *and* $f|C$ *is an immersion-germ;*
3) $j^2 f(0) \in \Sigma^{n-p+1,0}$;
4) f *can be expressed in* C^τ *local coordinates as*

$$f(u, x) = (u, -x_1^2 \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_{n-p+1}^2)$$

for some i *(*$i = 0$ *in case* $E = \mathbb{C}$*).*

Proof. This is (3.4) of [duPW1], except that condition (1′) was not stated there. An easy calculation shows that (4) implies (1′). On the other hand, if (1′) holds, then f must have corank 1, as shown in the proof of (3.4) of [duPW1]. If f is any map-germ of corank 1, then we can see from the normal form of (2.5) that

$$\dim_E E_n^p/(df(E_n^p) + f^* m_p E_n^p) = \dim_E E_n/(J(f) + f^* m_p E_n). \quad \square$$

Define E_C *to be* $E_n/J(f)$ *and* M_C *to be* $\operatorname{cok} df = E_n^p/df E_n^n$. For the next several results, we will restrict our attention to the *complex analytic case*: $f : \mathbb{C}^n, 0 \longrightarrow \mathbb{C}^p, 0$ is a complex analytic map-germ with $n \geq p$.

(2.10) Proposition. *(*$\mathbb{C}$-ω*).*

In all cases $\dim C \geq p - 1$. *If* $\dim C = p - 1$, *then* E_C *and* M_C *are Cohen-Macaulay as either* E_n *or* E_C *modules.*

Proof. In (1.13), let $\varphi = df$ and $k = p$ or 1. Then $I(\varphi) = J(f)$ and, by (1.10), $\operatorname{cod} C = \operatorname{depth}_{E_n}(J(f), E_n) \leq n - p + 1$. Thus $\dim C \geq p - 1$.

If $\dim C = p - 1$, (1.13.2) and (2.3) imply that $\operatorname{hd}_{E_n} E_C = \operatorname{hd}_{E_n}(M_C) = n - p + 1 = n - \dim E_C = n - \dim(E_n/\operatorname{ann}(M_C))$. Thus, by (1.7), E_C and M_C are Cohen-Macaulay as E_n modules and, by (1.2), as E_C modules. $\square$

(2.11) Theorem. *(*$\mathbb{C}$-ω*).* f *has non-degenerate critical set if, and only if,* $j^2 f \in \Sigma_t^{n-p+1}$ *off a codimension* ≥ 1 *subvariety of* C.

Proof. By (2.10), $\dim C \geq p - 1$. But $\dim C = p - 1$ if either f has non-degenerate critical set (by definition) or if $j^2 f \in \Sigma_t^{n-p+1}$ generically (for if $\dim C > p - 1$, there would be points of C of dimension greater than $p - 1$ at which $j^2 f \in \Sigma_t^{n-p+1}$, contradicting (2.6)). Thus, in either case, E_C is Cohen-Macaulay by (2.10). Thus f has non-degenerate critical set if, and only if, $j^2 f \in \Sigma_t^{n-p+1}$ generically, by (1.18) and (2.6). $\square$

(2.12) Proposition. *($\mathbb{C}$-ω). Suppose f has non-degenerate critical set. Then f is generically finite-to-one on its critical set if, and only if, $C - \Sigma^{n-p+1,0}(f)$ is a codimension ≥ 1 subvariety of C.*

Proof. We will prove "only if", "if" being trivial.

Using (2.10.1), it is easy to see that $C - \Sigma^{n-p+1,0}(f)$ is an analytic variety.

Let F be any representative of f. Since $\mathcal{J}(F)$ and $\mathcal{I}(C(F))$ are finitely generated, hence coherent sheaves of ideals, the set of points at which their stalks are unequal is a subvariety. The set of points at which $\dim C(f)_x > p-1$ is also a subvariety. Thus the set of points at which F_x has degenerate critical set is a subvariety. Thus there is an open neighborhood of 0 at each point x of which F_x has non-degenerate critical set; we henceforth restrict F to this neighborhood.

Now assume f is generically finite-to-one on its critical set. We can assume $F|C(F)$ has discrete fibers off a codimension ≥ 1 subvariety of $C(F)$. If the Proposition were not true, we could find an open subset U of $C(F) - \Sigma^{n-p+1,0}(F)$ such that U is a manifold of dimension $p-1$. By (2.9), $F|U$ has rank less than $p-1$ at each point of U. Shrinking U if necessary, we may assume $F|U$ has constant rank $k < p-1$. But then $F|U$ is locally equivalent to a projection to E^k, so the fibers have positive dimension, contradicting the hypothesis. □

If $n < p$, a similar argument shows that f is generically finite-to-one on its critical set if, and only if, f is generically an immersion.

(2.13) Lemma. *($\mathbb{C}$-ω). f has non-degenerate critical set with E_C normal if, and only if, $j^2 f \in \Sigma_t^{n-p+1}$ off a codimension 2 subvariety of C.*

Proof. Follows from (2.11), (2.10), (1.21) and (2.6). □

An immediate consequence is:

(2.14) Proposition. *($\mathbb{C}$-ω). Suppose f is FST. Then f is a CN if, and only if, $j^2 f \in \Sigma_t^{n-p+1}$ off a codimension 2 subvariety of C and $f|C$ is generically one-to-one.*

The $\mathbb{C}$-ω case of Theorem (0.8) is merely a rephrasing of this Proposition.

For the next several results, we need to consider germs at a finite set. Let $f : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, \{y\})$, $n \geq p$, $S = \{x_1, \ldots, x_s\}$, be analytic. Suppose f is FST, that is $f|C$ is finite-to-one, where $C = C(f)$. Let f_i be the germ of f at x_i. Then $C_i = C(f_i)$ and $D = f(C)$ are variety-germs of dimension $p-1$. Let E_C denote $\mathcal{E}_S/\mathcal{J}(f)_S$ and let E_p denote $\mathcal{E}_y$. Then E_C is a finite E_p module, so has a Fitting ideal $F_0(E_C) \subset E_p$ (see the definition preceding (1.14)), and $V(F_0(E_C)) = D$. Let $E_D = E_p/F_0(E_C)$. Similarly, $M_D = E_p/F_0(M_C)$.

(2.15) Proposition. *($\mathbb{C}$-ω). Under the assumptions and with the notation of the above paragraph:*

1) E_D is Cohen-Macaulay and $F_0(E_C)$ is generated by some $\varphi = \varphi_1 \cdots \varphi_s$, $\varphi_i \in m_p^{r_i}$, where $r_i = \dim_{\mathbb{C}} \mathcal{E}_{x_i}/(\mathcal{J}(f)_{x_i} + f_i^* m_p \mathcal{E}_{x_i})$.

2) M_D is Cohen-Macaulay and $F_0(M_C)$ is generated by some $\varphi = \varphi_1 \cdots \varphi_s$, $\varphi_i \in m_p^{r'_i}$, where $r'_i = \dim_{\mathbb{C}} \mathcal{E}_{x_i}^p/(df\mathcal{E}_{x_i}^n + f_i^* m_p \mathcal{E}_{x_i}^p) = \mathcal{K}_e$-cod$(f_i)$.

Proof. We will prove (1); the proof of (2) is identical. By (2.10), E_{C_i} is Cohen-Macaulay as an E_{C_i} module, i.e.,

$$\operatorname{depth}_{E_{C_i}} E_{C_i} = \dim E_{C_i} = p - 1.$$

By (1.2), $\operatorname{depth}_{E_p} E_{C_i} = p - 1$. If $\alpha \in E_p$ is a nonzero function germ which vanishes on $f(C_i)$, then $\alpha \circ f$ vanishes on C_i, so $\alpha^k \circ f \in J(f_i)$ for some k by the Nullstellensatz. Thus $\operatorname{ann}_{E_p} E_{C_i} \neq 0$.

Suppose $r_i = \dim_{\mathbb{C}} \mathcal{E}_{x_i}/(J(f_i) + f_i^* m_p \mathcal{E}_{x_i})$. By the Malgrange form of the Preparation Theorem, there exist $g_{i,1}, \dots, g_{i,r_i} \in \mathcal{E}_{x_i}$ which generate E_{C_i} as an $f_i^* E_p$ module. Then $\pi_i : E_p^{r_i} \longrightarrow E_C$ sending $(\alpha_1, \dots, \alpha_{r_i})$ to $\sum_{j=1}^{r_i} (\alpha_j \circ f_i) g_{i,j}$ is surjective. By the proof of (1.14), there is an E_p module map Φ_i such that

$$0 \longrightarrow E_p^{r_i} \xrightarrow{\Phi_i} E_p^{r_i} \xrightarrow{\pi_i} E_{C_i} \longrightarrow 0$$

is exact, and $\varphi_i = \det \Phi_i$ generates $F_0(E_{C_i})$. Now $\overline{g}_{i,1}, \dots, \overline{g}_{i,r_i}$ form a basis of $\mathcal{E}_{x_i}/(J(f_i) + f_i^* m_p \mathcal{E}_{x_i})$ as a $\mathbb{C}$ vector space. Suppose $\pi_i(\alpha_1, \dots, \alpha_{r_i}) = \Sigma(\alpha_j \circ f_i) g_{i,j} = 0$. Then the linear independence of the $\overline{g}_{i,j}$ implies that each $\alpha_j(0) = 0$. Thus each entry of Φ_i (interpreted as a matrix) is in m_p, so $\det \Phi_i \in m_p^{r_i}$.

Extend $g_{i,j}$ to be the germ of the identically zero function at each $x_k \neq x_i$. Relabel $g_{i,1}, \dots, g_{s,r_s}$ as $h_1, \dots, h_r$, $r = r_1 + \dots + r_s$. The $\overline{h}_i$'s form a basis of $\mathcal{E}_S/(J(f) + f^* m_p \mathcal{E}_S)$ as a $\mathbb{C}$ vector space. Clearly we have an exact sequence

$$0 \longrightarrow E_p^r \xrightarrow{\Phi} E_p^r \xrightarrow{\pi} E_C \longrightarrow 0 \quad ,$$

where Φ is the direct sum of Φ_i's. Thus $\varphi = \det \Phi = \varphi_1 \cdots \varphi_S \in m_p^r$ and generates $F_0(E_C)$ as desired. □

From (2.15) and (2.9), we get:

(2.16) Corollary. *($\mathbb{C}$-ω). Suppose $f : U \subset \mathbb{C}^n \longrightarrow \mathbb{C}^p$ is finite-to-one on $C(f)$, $n \geq p$. Let $D = f(C(f))$. Then E_{D_y} is reduced and D_y is nonsingular if, and only if, $f^{-1}(y) \cap C(f) = \{x\}$ for some x and $j^2 f(x) \in \Sigma^{n-p+1,0}$.*

From (1.18), (2.15) and (2.16), we immediately get:

(2.17) Corollary. *($\mathbb{C}$-ω). Suppose f is FST. Then f is a CS if, and only if, E_D is reduced.*

Note that E_D is not reduced for the map in Example (2.2.3).

Next we take up the comparison of the $\mathbb{C}$-ω and $\mathbb{R}$-ω cases. Suppose $f : (R^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, $n \geq p$, is analytic. Let $f_{\mathbb{C}} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$ be its complexification. Let $C_{\mathbb{C}}$ denote the complexification of $C = C(f)$ (see for example [N]). Let E_n denote the analytic germs from $(\mathbb{R}^n, 0)$ to $\mathbb{R}$ and $E_n(\mathbb{C})$ those from $(\mathbb{C}^n, 0)$ to $\mathbb{C}$.

(2.18) Proposition. *($\mathbb{R}$-ω). f has non-degenerate critical set if, and only if, $f_{\mathbb{C}}$ has non-degenerate critical set and $C_{\mathbb{C}} = C(f_{\mathbb{C}})$ if, and only if,* $\operatorname{cod}(C - \Sigma_t^{n-p+1}(f)) \geq 1$ *and $C_{\mathbb{C}} = C(f_{\mathbb{C}})$.*

Proof. Clearly $J(f_{\mathbb{C}}) = J(f)E_n(\mathbb{C})$ and $I(C_{\mathbb{C}}) = I(C)E_n(\mathbb{C})$. Suppose $I(C) = J(f)$. Then $I(C_{\mathbb{C}}) = J(f_{\mathbb{C}})$, so $C_{\mathbb{C}} = C(f_{\mathbb{C}})$ and $I(C(f_{\mathbb{C}})) = J(f_{\mathbb{C}})$.

Conversely, suppose $C_{\mathbb{C}} = C(f_{\mathbb{C}})$ and $I(C(f_{\mathbb{C}})) = J(f_{\mathbb{C}})$. Then

$$I(C)E_n(\mathbb{C}) = I(C(f_{\mathbb{C}})) = J(f_{\mathbb{C}}) = J(f)E_n(\mathbb{C}).$$

Thus

$$\begin{aligned} I(C) &= (I(C)E_n(\mathbb{C})) \cap E_n(\mathbb{R}) \\ &= (J(f)E_n(\mathbb{C})) \cap E_n(\mathbb{R}) = J(f) \ . \end{aligned}$$

Of course, $\dim C = \dim C(f_{\mathbb{C}})$ if $C_{\mathbb{C}} = C(f_{\mathbb{C}})$. Thus f has non-degenerate critical set if, and only if, $f_{\mathbb{C}}$ has non-degenerate critical set and $C_{\mathbb{C}} = C(f_{\mathbb{C}})$.

Again suppose $C_{\mathbb{C}} = C(f_{\mathbb{C}})$. Then, for each irreducible component V of $C(f_{\mathbb{C}})$, $V \cap \mathbb{R}^n$ is an irreducible component of C; for each irreducible component of C, $C_{\mathbb{C}}$ is an irreducible component of $C(f_{\mathbb{C}})$. Thus $\operatorname{cod}(C - \Sigma_t^{n-p+1}(f)) = 0$ if, and only if, $C - \Sigma_t^{n-p+1}(f)$ contains an irreducible component of C if, and only if, $C(f_{\mathbb{C}}) - \Sigma_t^{n-p+1}(f_{\mathbb{C}})$ contains an irreducible component of $C(f_{\mathbb{C}})$ if, and only if, $\operatorname{cod}(C(f_{\mathbb{C}}) - \Sigma_t^{n-p+1}(f)) = 0$ if, and only if, $f_{\mathbb{C}}$ has degenerate critical set. □

(2.19) Proposition. *($\mathbb{R}$-ω).*

1) *Suppose f has non-degenerate critical set. Then f is generically finite-to-one on its critical set if, and only if, $C - \Sigma^{n-p+1,0}(f)$ is a codimension ≥ 1 subvariety of C.*
2) *f is generically finite-to-one on its critical set if, and only if, $f_{\mathbb{C}}$ is generically finite-to-one on its critical set and $C_{\mathbb{C}} = C(f_{\mathbb{C}})$.*
3) *f is a CN if, and only if, $f_{\mathbb{C}}$ is a CN and $C_{\mathbb{C}} = C(f_{\mathbb{C}})$.*

Proof. Suppose f has non-degenerate critical set; so C has dimension $p-1$ and $C_{\mathbb{C}} = C(f_{\mathbb{C}})$.

1) Let F be a representative of f such that $F_{\mathbb{C}}$ has non- degenerate critical set at each point. The regular points of dimension $p-1$ of $C(F)$ form a nonempty open subset of $C(F)$. At each such point x, $C(F_x)_{\mathbb{C}}$ is a $p-1$ dimensional manifold contained in $C(F_{\mathbb{C}})_x$, which is also $p-1$ dimensional. The singular set of $C(F_{\mathbb{C}})$ is of dimension less than $p-1$, so there exist points x such that $C(F_x)$ and $C(F_{\mathbb{C}})_x$ are $p-1$ dimensional manifolds; at such points $C(F_x)_{\mathbb{C}} = C(F_{\mathbb{C}})_x$. By (2.18), F_x has non-degenerate critical set at these points. Now one can repeat the proof of (2.12).

2) Let S (respectively $S^{\mathbb{C}}$) be the subvariety of C (respectively $C_{\mathbb{C}}$) of non-$\Sigma^{n-p+1,0}$ points. Certainly $S^{\mathbb{C}} \cap \mathbb{R}^n = S$ (does $S^{\mathbb{C}} = S_{\mathbb{C}}$?). Thus $f_{\mathbb{C}}$ is not generically finite-to-one on its critical set if, and only if, $S^{\mathbb{C}}$ contains an irreducible component of $C_{\mathbb{C}}$ if, and only if, S contains an irreducible component of C if, and only if, f is not generically finite-to-one on its critical set.

3) By definition, $f|C : C \longrightarrow D$ is a $\mathbb{R}$-ω normalization if, and only if, $(f|C)_{\mathbb{C}} : C_{\mathbb{C}} \longrightarrow D_{\mathbb{C}}$ is a $\mathbb{C}$-ω normalization. If $C_{\mathbb{C}} = C(f_{\mathbb{C}})$, then this holds if, and only if, $f_{\mathbb{C}} \mid C(f_{\mathbb{C}}) : C(f_{\mathbb{C}}) \longrightarrow D(f_{\mathbb{C}})$ is a normalization. The result then follows from (2.18). □

(2.20) Corollary. *If $f_{\mathbb{C}}$ is a CS and $C(f)_{\mathbb{C}} = C(f_{\mathbb{C}})$, then f is a CS.*

The converse of (2.20) does not hold, as Example (2.21.2) shows.

(2.21) Examples

1) $f(u,z) = (u, z^4 + u^2 z^2)$ is one-to-one on $C(f)$ and has only fold singularities on $C(f) - \{0\}$; in fact $f_{\mathbb{C}}$ is *FST-CS*; but $C(f_{\mathbb{C}}) \neq C(f)_{\mathbb{C}}$, so $J(f) \neq I(C(f))$ (f has *degenerate critical set*).
2) $f(x,y) = (x^2, y^3 + xy)$ is a CS and $C(f)_{\mathbb{C}} = C(f_{\mathbb{C}})$, but $f_{\mathbb{C}}$ is not a CS.

Now we will consider the C^∞ case. $\mathcal{E}$ denotes the sheaf of germs of C^∞ functions on $\mathbb{R}^n$. A differentiable sheaf $\mathcal{M}$ over an open set $\Omega \subseteq \mathbb{R}^n$ (i.e., a sheaf of $\mathcal{E}$-modules with unit) is *quasi-flabby* (introduced as *quasi-flasque* in [To]) if, for every open $U \subset \Omega$, the map

$$\varphi : \mathcal{M}(\Omega) \otimes_{\mathcal{E}(\Omega)} \mathcal{E}(U) \longrightarrow \mathcal{M}(U)$$

defined by

$$\sum m_i \otimes f_i \longrightarrow f_i r(m_i)$$

is an isomorphism, where $r : \mathcal{M}(\Omega) \longrightarrow \mathcal{M}(U)$ is the "restriction" map (see Chapter V Section 6 of [To]).

First we will observe that φ is always injective. Suppose $\varphi(\Sigma m_i \otimes f_i) = 0$, i.e., $\Sigma f_i r(m_i) = 0$. By V.6.1 of [To], there is a C^∞ function α on Ω which is positive on U and infinitely flat on $\Omega - U$ such that αf_i is C^∞ on Ω and infinitely flat on $\Omega - U$. Thus $\Sigma(\alpha f_i) m_i = 0$ on Ω. Thus,

$$\begin{aligned}\sum m_i \otimes f_i &= \sum m_i \otimes (\alpha f_i)\frac{1}{\alpha} \\ &= \sum m_i(\alpha f_i) \otimes \frac{1}{\alpha} = 0 \otimes \frac{1}{\alpha} = 0 \quad .\end{aligned}$$

(2.22) Lemma. *Let X be a subset of Ω and let $\mathcal{I}$ be the sheaf of C^∞ germs which vanish on X. Then $\mathcal{I}$ is quasi-flabby. If $\mathcal{I}_x$ is generated by the germs at x of C^∞ functions $f_1, \ldots, f_k$, then there is a neighborhood U of x such that $(f_1)_y, \ldots, (f_k)_y$ generate $\mathcal{I}y$ at each $y \in U$ and $f_1 \mid U, \ldots, f_k \mid U$ generate $\mathcal{I}(U)$.*

Proof. To prove that $\mathcal{I}$ is quasi-flabby, it is enough to show that φ is surjective. Pick $f \in \mathcal{I}(U)$. By V.6.1 of [To], there is a C^∞ function α which is positive on U and infinitely flat on $\Omega - U$ such that $g = \alpha f$ is C^∞ on Ω and infinitely flat on $\Omega - U$. Of course, g is in $\mathcal{I}(\Omega)$. On U, $f = g/\alpha = \varphi(g \otimes (1/\alpha))$.

The second claim follows from V.6.4 of [To] and a partition of unity argument. □

(2.23) Proposition. *Suppose $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, $n \geq p$, is a C^∞ map-germ which has non-degenerate critical set. Then $\Sigma_t^{n-p+1}(f)$ is dense in C.*

Proof. By (2.22), there is an open neighborhood U of 0 and a representative F of f on U such that $J(F) = I(C(F))$ and such that

(*) $$J(F_y) = I(C(F_y)) \text{ for all } y \in U \ .$$

Since $I(C(F))$ is closed and finitely generated, it is a Lojasiewicz ideal (see V.4.4 of [To]), so the regular points of $C(F)$ are dense (see the proof of V.4.6 of [To]).

By (2.7), these regular points of $C(F)$ have dimension $\geq p-1$. By assumption, $C(F)_0$ has dimension $p-1$, so, shrinking U if necessary, we may assume these regular points are all of dimension $p-1$. By (2.6), $j^2F(x) \in \Sigma_t^{n-p+1}$ at these regular points. □

Let $I^\infty(C)$ (respectively $I^\omega(C)$) denote the ideal of C^∞ (respectively real analytic) function-germs vanishing on C; we use a similar notation for $J(f)$, E_n and the corresponding sheaves. An analytic variety-germ C is said to be coherent if it has a representative C' such that $\mathcal{I}^\omega(C')$ is finitely generated.

(2.24) Proposition. *Suppose $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, $n \geq p$, is C^∞ equivalent to an analytic germ g. The following are equivalent:*

1) *f is C^∞ with non-degenerate critical set;*
2) *g is $\mathbb{R}$-ω with non-degenerate critical set and $C(g)$ is coherent;*

For G any representative of g,

3) *G_x is $\mathbb{R}$-ω with non-degenerate critical set for all x sufficiently near 0;*

4a) *$\Sigma_t^{n-p+1}(g)$ is dense in $C(g)$ and*

b) *$(C(G_x))_{\mathbb{C}} = C((G_x)_{\mathbb{C}})$ for all x sufficiently near 0.*

Proof. First note that, for $C = C(g)$,

$$\begin{aligned} I^\omega(C)E_n^\infty &\subseteq I^\infty(C) \\ &\supseteq J^\infty(g) \\ &= J^\omega(g)E_n^\infty \\ &\subseteq I^\omega(C)E_n^\infty \quad . \end{aligned}$$

By a theorem of Malgrange (VI.3.10 of [Mal]; or see VI.4.2 of [To]), C is coherent if, and only if, $I^\omega(C)E_n^\infty = I^\infty(C)$. By I.4.9 of [To] and V.1.12 of [Mal], $IE_n^\infty \cap E_n^\omega = I$ for any ideal $I \subset E_n^\omega$. Thus $J^\omega(g)E_n^\infty = I^\omega(C)E_n^\infty$ if, and only if, $J^\omega(g) = I^\omega(C)$. Thus $I^\infty(C) = J^\infty(g)$ if, and only if, $I^\omega(C) = J^\omega(g)$ and $C(g)$ is coherent. Thus f is C^∞ with non-degenerate critical set if, and only if g is C^∞ with non-degenerate critical set if, and only if, g is $\mathbb{R}$-ω with non-degenerate critical set and $C(g)$ is coherent.

By definition, $C(g)$ is coherent if, and only if, $\mathcal{I}^\omega(C(G))$ is finitely generated (i.e. coherent) on a neighborhood of 0. Since $\mathcal{J}^\omega(G)$ is coherent, $\mathcal{I}^\omega(C(G)) = \mathcal{J}^\omega(G)$ on a neighborhood of U if, and only if, $\mathcal{I}^\omega(C(G))_0 = \mathcal{J}^\omega(G)_0$ and $\mathcal{J}^\omega(C(G))$ is coherent on a neighborhood of 0. Thus (2) is equivalent to (3).

It is immediate from (2.18) that (3) $\Rightarrow$ (4). On the other hand, (4a) implies that $\operatorname{cod}(C(G)_x - \Sigma_t^{n-p+1}(G_x)) \geq 1$ for all x sufficiently near 0. So (4) $\Rightarrow$ (3) by (2.18), also. □

(2.25) Example $f(u_1, u_2, u_3, z) = (u_1, u_2, u_3, z^3 - zu_1(u_2^2 + u_3^2))$ is a $\mathbb{R}$-ω CN; but C is not coherent; for $C(f)_{\mathbb{C}} \neq C(f_{\mathbb{C}})$ along the negative u_1-axis. So f does not have non-degenerate critical set in the C^∞ category.

For general results on when an ideal $I \subset E_n^\infty$ satisfies $I(V(I)) = I$, see [AL] and [Bo]. Some of the arguments in [Bo] are similar to those given here.

(2.26) Proposition. *Suppose $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, $n \geq p$, is C^∞ with non-degenerate critical set.*

1) *f is generically finite-to-one on its critical set if, and only if, $\Sigma^{n-p+1,0}(f)$ is dense in C*
2) *If f is a CS, then f is generically an embedding on C.*

Proof. Since f has non-degenerate critical set, $\Sigma_t^{n-p+1}(f)$ is dense in C. Now one can mimic the proof of (2.12) to prove (1).

f a CS implies that $f|C$ is generically one-to-one. By (1), $f|C$ is generically an immersion. Thus $f|C$ is generically an embedding. □

(2.27) Proposition. *Suppose $f \in E_n^p$ is equivalent to an analytic map-germ g whose complexification is a CN. Suppose $\dim C(G)_x = \min\{p-1, n\}$, for all x sufficiently near 0, for any representative G of g. Then f is a C^∞ CN.*

Proof. That $f|C(f)$ is a normalization follows immediately from the definition. We are done in case $n < p$, so assume $n \geq p$.

$C(G_{\mathbb{C}})_x$ is normal for all x sufficiently near 0, so, in particular, is irreducible. Thus $\dim C(G)_x = p-1$ implies $C(G_x)_{\mathbb{C}} = C(G_{\mathbb{C}})_x$. But $g_{\mathbb{C}}$ a CN implies $(G_{\mathbb{C}})_x$ has non-degenerate critical set for all x sufficiently near 0. By (2.18), G_x is $\mathbb{R}$-ω with non-degenerate critical set for all x sufficiently near 0. By (2.24), f has non-degenerate critical set in the C^∞ category. Thus f is a C^∞ CN. □

§3. Discriminant matrices.

We begin this section with some recipes for calculating discriminants. It is implicit in the arguments of §§1,2 that the reduced equation of the discriminant of a CS, at least in the $\mathbb{C}$-analytic case, can be obtained by taking the determinant of an appropriate square matrix. We make this explicit here. It will appear that the matrices can be constructed in a rather canonical way; we call them *discriminant matrices*. It seems that they encode information about the map-germ they are defined from in a rather more convenient form than the equation of the discriminant itself; we will conclude with a discussion of this, including results on $\mathcal{A}$-codimension.

Our first recipe is a concretization of (2.15) — this procedure can also be found in (2.2) of [MP], where a different argument is given.

(3.1) Proposition. *We adapt the notation of 2.15. Let $f : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, y)$ be a $\mathbb{C}$-analytic map-germ of finite singularity type, with $n+1 \geq p$. Let $\pi : \mathbb{C}^p \longrightarrow \mathbb{C}^{p-1}$ be the linear projection $\pi(y_1, \ldots, y_p) = (y_1, \ldots, y_{p-1})$.*

Suppose that $\alpha_1, \ldots, \alpha_s \in \mathcal{E}_S$ project to a $\mathbb{C}$-basis for

$$E_C/(\pi \circ f)^* m_{p-1} E_C.$$

Then, for $i = 1, \ldots, s$, $f_p \alpha_i$ can be written uniquely as a $(\pi \circ f)^* E_{p-1}$-linear combination of the $\alpha_1, \ldots, \alpha_s$ modulo $\mathcal{J}(f)_S$, say

$$f_p \alpha_j = \sum_{i=1}^{s} (\pi \circ f)^* v_{ij} \cdot \alpha_i.$$

Let V be the matrix whose entries are the v_{ij}, let I be the identity matrix and let $\delta = y_p I - V$. Then there is an exact sequence

$$0 \longrightarrow E_p^s \xrightarrow{\delta} E_p^s \xrightarrow{\pi} E_C \longrightarrow 0,$$

where $\pi(e_i)$ is the projection of α_i, $\{e_i\}$ the natural basis of E_p^s.

Moreover, the ideal $\langle \det \delta \rangle \subset E_p$ defines the discriminant locus; it is reduced if, and only if, f is a CS.

Proof. By the Preparation Theorem, $\alpha_1, \dots, \alpha_s$ form a basis for E_C as an E_{p-1}-module (with action induced by $(\pi \circ f)^*$. We claim that they form a free basis. For, arguing as in (2.15), we have that E_{C_i} is Cohen-Macaulay as an $\mathcal{E}_{x_i}$-module, so that

$$\operatorname{depth}_{\mathcal{E}_{x_i}} E_{C_i} = \dim E_{C_i} = p - 1,$$

since $\dim C(f) = p - 1$. Then, by (1.2),

$$\operatorname{depth}_{E_{p-1}} E_{C_i} = p - 1,$$

and so, by (1.4), it has homological dimension zero. Since E_C is the direct sum of the E_{C_i}, this completes the argument.

For $j = 1, \dots, s$, $f_p \alpha_j$ projects to an element of E_C, which can be written as an E_{p-1}-linear combination of the basis elements; so there exist $v_{ij} \in E_{p-1}$ with

$$(r_j) \qquad f_p \alpha_j = \sum_{i=1}^{s} (\pi \circ f)^* v_{ij} \cdot \alpha_i.$$

This representation is unique because the basis is free.

We claim that all E_p-relations amongst the α_i are E_p-linear combinations of the (r_j). To be more precise, let

$$R = \{\, (\phi_1, \dots, \phi_s) \in E_p^s : \sum_{j=1}^{s} (\phi_j \circ f) \alpha_j \in \mathcal{J}(f) \,\}.$$

This is clearly an E_p-module. We claim that

$$R = E_p \langle A_1, \dots, A_s \rangle,$$

where

$$A_i = y_p e_j - \sum_{i=1}^{s} v_{ij} e_i.$$

To see this, consider $\Phi = (\phi_1, \dots, \phi_s) \in R$. For $j = 1, \dots, s$, we can write

$$\phi_j = \xi_j + y_p \eta_j,$$

with $\xi_j \in E_{p-1}$ and $\eta_j \in E_p$. As above, we can write, for $j = 1, \ldots, s$,

$$f^*\eta_j \cdot \alpha_j = \sum_{j=1}^{s} (\pi \circ f)^* w_{ij} \cdot \alpha_i \bmod \mathcal{J}(f),$$

with $w_{ij} \in E_{p-1}$. Set

$$N_j = \eta_j e_j - \sum_{i=1}^{s} w_{ij} e_i;$$

by the above, we have $N_j \in R$.

Thus

$$\begin{aligned}
\Phi &= \sum_{j=1}^{s} (\xi_j + y_p \eta_j) e_j \\
&= \sum_{j=1}^{s} \xi_j e_j + y_p \sum_{j=1}^{s} (N_j + \sum_{i=1}^{s} w_{ij} e_i) \\
&= \sum_{j=1}^{s} \xi_j e_j + y_p \sum_{j=1}^{s} N_j + \sum_{j=1}^{s} \sum_{i=1}^{s} w_{ij} (A_i + \sum_{k=1}^{s} v_{ik} e_k) \\
&= \sum_{k=1}^{s} (\xi_k + \sum_{j=1}^{s} \sum_{i=1}^{s} w_{ij} v_{ik}) e_k + y_p \sum_{j=1}^{s} N_j + \sum_{j=1}^{s} \sum_{i=1}^{s} w_{ij} A_i.
\end{aligned}$$

Since $\Phi,\ A_i,\ N_i$ are in R, we have that

$$\sum_{k=1}^{s} (\pi \circ f)^* (\xi_k + \sum_{j=1}^{s} \sum_{i=1}^{s} w_{ij} v_{ik}) \cdot \alpha_k \in \mathcal{J}(f)_S.$$

Since the α_k are a free E_{p-1}-basis for E_C, the coefficients on the left hand side must be zero, so

$$\Phi = y_p \sum_{j=1}^{s} N_j + \sum_{j=1}^{s} \sum_{i=1}^{s} w_{ij} A_i.$$

Since $\Phi \in R$ was chosen arbitrarily, this shows

$$R \subset E_p \langle A_1, \ldots, A_s \rangle + m_p R.$$

R is a finitely generated E_p-module (e.g. because it is the stalk of a morphism of coherent E_p-sheaves), so, by Nakayama's Lemma, $R \subset E_p \langle A_1, \ldots, A_s \rangle$, as required.

It follows that

$$E_p^s \xrightarrow{\delta} E_p^s \xrightarrow{\pi} E_C \to 0$$

is an exact sequence of E_p-modules, where $\delta e_j = A_j$. Thus $\det \delta$ generates $(F_0)_{E_p}(E_C)$ and so defines the discriminant locus.

We need to show that δ is injective. If $\ker\delta \neq 0$, then there exists a not-identically-zero $\xi \in \ker\delta \subset E_p^s$. There is a Zariski open set U on which $\xi \neq 0$. Since δ is square, $\det\delta = 0$ on U, hence everywhere. Thus $(F_0)_{E_p}(E_C) = \{0\}$, implying that the discriminant is all of $\mathbb{C}^p$, which is false. Thus δ is injective.

By (2.17), E_D is reduced if, and only if, f is a CS. □

Our next result takes a more module-theoretic point of view. Where we previously have viewed the critical space as given by the ring $E_C = \mathcal{E}_S/\mathcal{J}(f)_S$, we here view it as given in some sense by the $\mathcal{E}_S$-module $M_C = \mathcal{E}_S^p/df(\mathcal{E}_S^n)$.

(3.2) Proposition. *Let $f : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, y)$ be a $\mathbb{C}$-analytic map-germ of finite singularity type, with $n \geq p$. Let $\pi : \mathbb{C}^p \longrightarrow \mathbb{C}^{p-1}$ be the linear projection $\pi(y_1, \dots, y_p) = (y_1, \dots, y_{p-1})$.*

Suppose that $\beta_1, \dots, \beta_t \in \mathcal{E}_S^p$ project to a $\mathbb{C}$-basis for

$$M_C/(\pi \circ f)^* m_{p-1} M_C.$$

Then, for $i = 1, \dots, t$, $f_p\beta_i$ can be written uniquely as a $(\pi \circ f)^* E_{p-1}$-linear combination of the $\beta_1, \dots, \beta_t$ modulo $df(\mathcal{E}_S^p)$, say

$$f_p\beta_j = \sum_{i=1}^{s} (\pi \circ f)^* w_{ij} \cdot \beta_i.$$

Let W be the matrix whose entries are the w_{ij}, let I be the identity matrix and let $\Delta = y_p I - W$. Then there is an exact sequence

$$0 \longrightarrow E_p^t \xrightarrow{\Delta} E_p^t \xrightarrow{\pi} M_C \longrightarrow 0,$$

where $\pi(e_i)$ is the projection of β_i, $\{e_i\}$ the natural basis of E_p^t.

Moreover, the ideal $\langle \det\Delta \rangle \subset E_p$ defines the discriminant locus; it is reduced if, and only if, f is a CS.

Proof. The argument is very similar to that for (3.1). To get started, we note that, f being of finite singularity type, $C(f)$ is of dimension $p-1$, and so E_{C_i} is too. Thus by (1.13), setting $M_{C_i} = \mathcal{E}_{x_i}^p/df(\mathcal{E}_{x_i}^n)$, we have $\operatorname{hd}_{\mathcal{E}_{x_i}} M_{C_i} = n - p + 1$. Since $\dim \mathcal{E}_{x_i} = n$, it follows from (1.4) that $\operatorname{depth}_{\mathcal{E}_{x_i}}(M_{C_i}) = p - 1$. But $\dim M_{C_i} = \dim \mathcal{E}_{x_i}/\operatorname{ann} M_{C_i} = p - 1$, and so M_{C_i} is a Cohen-Macaulay $\mathcal{E}_{x_i}$-module. Taking direct sums we see that M_C is a Cohen-Macaulay $\mathcal{E}_C$-module, of depth $p-1$.

The argument that

$$0 \longrightarrow E_p^t \xrightarrow{\Delta} E_p^t \xrightarrow{\pi} M_C \longrightarrow 0,$$

is exact is now exactly as in (3.1). We have then that $(F_0)_{E_p}(M_C) = \langle \det\Delta \rangle$, so

$$\begin{aligned}(V(\det\Delta), y) &= (\operatorname{supp}((F_0)_{E_p}(M_C)), y)\\ &= (f \operatorname{supp}((F_0)_{\mathcal{E}_S}(M_C)), y)\\ &= (fC(f), y) = (D, y),\end{aligned}$$

as required.

Suppose now that f is a CS. Then, by (2.12),

$$D' = \{\, y' \in D : f^{-1}y' \cap C = \{x\}, \text{ and } f_x \text{ is of fold type} \,\}$$

is dense in D.

Now for any point $y' \in D$ such that all points of $S_{y'} = f^{-1}y' \cap C$ have kernel rank $n - p + 1$,

$$\mathcal{E}_{S_{y'}}/J(f)_{S_{y'}} \cong \mathcal{E}^p_{S_{y'}}/df(E^n_{S_{y'}})$$

as $\mathcal{E}_{S_{y'}}$-, hence also as $\mathcal{E}_{y'}$-, modules, and so have the same 0^{th} Fitting ideals. In particular, then, for $y' \in D'$,

$$\langle \det \Delta \rangle = (F_0)_{\mathcal{E}_{y'}}(M_C) \cong (F_0)_{\mathcal{E}_{y'}}(E_C),$$

and is reduced since a fold-germ is certainly a CS.

Thus, by (1.18), $\det \Delta$ is reduced.

We defer proof of the converse until after (3.4). □

We now compare the hypotheses and conclusions of (3.1) and (3.2).

The first point to make is that the finite dimensionality hypotheses implicit in the choice of $\mathbb{C}$-bases need not hold for the standard coordinates in $\mathbb{C}^p$. However, for any given choice of coordinates for $(\mathbb{C}^p, y)$, they either both hold or both fail; and they do hold for a generic choice of coordinates, as the following lemma shows.

(3.3) Lemma. *The following are equivalent:*

(1) $\dim_{\mathbb{C}} E_C/(\pi \circ f)^* m_{p-1} E_C < \infty$;
(2) $\dim_{\mathbb{C}} M_C/(\pi \circ f)^* m_{p-1} M_C < \infty$;
(3) $\pi \circ f|(C, S)$ *is finite-to-one;*
(4) $f|(C, S)$ *and* $\pi|(D, y)$ *are finite-to-one;*
(5) f *and* $\pi \circ f$ *are of finite singularity type.*

Proof. Recall that a map-germ g being of finite singularity type is equivalent to $g|C(g)$ being finite-to-one. Thus (3) implies (5) because $C(\pi \circ f) \subset C(f)$. Clearly (3) and (4) are equivalent.

(3) ⇒ (2): Since the direct image of a coherent sheaf is coherent, $(\pi \circ f)_* M_C$ is coherent. In particular, its stalk at $\pi(y)$ is a finitely-generated E_{p-1}-module; by the Preparation Theorem this is equivalent to (2).

(2) ⇒ (1): If (2) holds, then

$$m^k_S \mathcal{E}^p_S \subset df(\mathcal{E}^n_S) + (\pi \circ f)^* m_{\pi(y)} \mathcal{E}^p_S$$

for some $k < \infty$. Thus, for any $\alpha_1, \ldots, \alpha_p \in m^k_S$ we can write the diagonal matrix with $\alpha_1, \ldots, \alpha_p$ down the diagonal as $df \cdot A + B$, with $A \in \mathcal{E}^n_S$ and $B \in (\pi \circ f)^* m_{\pi(y)} \mathcal{E}^p_S$. Taking determinants, we see that

$$\alpha_1 \cdot \cdots \cdot \alpha_p \in J(f) + (\pi \circ f)^* m_{\pi(y)} \mathcal{E}_S.$$

Thus

$$m_S^{kp} \subset J(f) + (\pi \circ f)^* m_{\pi(y)} \mathcal{E}_S,$$

and (1) holds.

(1) ⇒ (3): (1) implies that $J(f) + (\pi \circ f)^* m_{\pi(y)} \mathcal{E}_S$ contains a power of m_S, so that

$$S \supset V(J(f) + (\pi \circ f)^* m_{\pi(y)} \mathcal{E}_S) = C \cap (\pi \circ f)^{-1} \pi(y).$$

But this implies $\pi \circ f | C(f)$ is finite-to-one, as required.

(5) ⇒ (4): (We thank C. T. C. Wall for showing us this argument).

Note that if $x \in C(f) - C(\pi \circ f)$, then $(df_1)_x, \ldots, (df_p)_x$ are not linearly independent, but $(df_1)_x, \ldots, (df_{p-1})_x$ are; thus $(df_p)_x$ is a linear combination of $(df_1)_x, \ldots, (df_{p-1})_x$.

Suppose (4) fails, so that $\pi | D(f)$ is not finite-to-one. Since $K_\pi = \pi^{-1}\pi(y)$ is of dimension one, this means

$$K_\pi \subset D(f) = f(C(f)).$$

Thus there is a non-trivial analytic curve-germ $\phi : (\mathbb{C}, 0) \longrightarrow (C(f), S)$ with $\pi \circ f \circ \phi = 0$, whence

$$(df_1)_{\phi(t)}(\frac{d\phi}{dt}) = 0, \ldots, (df_{p-1})_{\phi(t)}(\frac{d\phi}{dt}) = 0, \text{ for } t \in (\mathbb{C}, 0).$$

If also $(df_p)_{\phi(t)}(\frac{d\phi}{dt}) = 0$ for all $t \in (\mathbb{C}, 0)$, then we would have $\frac{d}{dt}(f \circ \phi)_t = 0$, so $f \circ \phi$ would be constant, so that $\operatorname{im} \phi \subset C(f) \cap f^{-1}(\phi(0))$, contradicting the fact that f is of finite singularity type. So $(df_p)_{\phi(t)}(\frac{d\phi}{dt}) \neq 0$ for almost all $t \in (\mathbb{C}, 0)$, and so $(df_p)_{\phi(t)}$ cannot be a linear combination of $(df_1)_{\phi(t)}, \ldots, (df_{p-1})_{\phi(t)}$ for such t. Hence, by our first remark, since $\phi(t) \in C(f)$, $\phi(t) \in C(\pi \circ f)$ for such t. But $\phi(t) \in (\pi \circ f)^{-1}(\pi(y))$ for all $t \in (\mathbb{C}, 0)$, so $\pi \circ f$ cannot be of finite singularity type, and (5) fails □

For fixed f and π, we now attempt to compare the δ and Δ of (3.1) and (3.2). In the case where f is of corank one at each point of S (and $n \geq p$), M_C and E_C are isomorphic as $\mathcal{E}_S$-modules, hence as E_p-modules, and so δ and Δ are "essentially the same"; that is, up to change of basis. (We have already used this isomorphism in the proof of (3.2); perhaps it is worth being more explicit: by appropriate choice of coordinates, f_{x_i} can be put into the "linearly adapted" form

$$f(u_1, \ldots, u_{p-1}, x_1, \ldots, x_{n-p+1}) = (u_1, \ldots, u_{p-1}, g(u, x)),$$

so that $df(\mathcal{E}_{x_i}^n)$ is generated by the columns of the Jacobian matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial g}{\partial u_1} & \ldots & \frac{\partial g}{\partial u_{p-1}} & \frac{\partial g}{\partial x_1} & \ldots & \frac{\partial g}{\partial x_{n-p+1}} \end{bmatrix},$$

while $J(f) = \langle \frac{\partial g}{\partial x_1} \ldots \frac{\partial g}{\partial x_{n-p+1}} \rangle$. It is easy to see that the $\mathcal{E}_{x_i}$-injection $\mathcal{E}_{x_i} \longrightarrow \mathcal{E}_{x_i}^p$ given by the inclusion in the last coordinate induces an $\mathcal{E}_{x_i}$-module isomorphism $E_{C_i} \cong M_{C_i}$. The result for general S follows by taking direct sums.)

However, when f is of corank greater than one at some point of S, E_C and M_C are not isomorphic as $\mathcal{E}_S$-modules, since the minimal number of generators required is different: if $S = \{x\}$ and f has corank k at x, then M_C requires k generators, but E_C requires only 1 generator, namely the constant function 1.

Neither are E_C and M_C isomorphic as E_p-modules in general. Again, we can sometimes see this simply by counting generators. For example, for the germ $f : (x, y) \mapsto (xy, x^2 + y^2)$, and so for any unfolding of this germ, we find $\dim_{\mathbb{C}}(E_C/f^* m_p E_C) = 3$, $\dim_{\mathbb{C}}(M_C/f^* m_p M_C) = 4$, which (by the Preparation Theorem) gives the numbers of E_p-generators for E_C, M_C respectively. However, there are many situations where the minimum number of generators involved is the same; this is the case whenever f is an (unfolding of) a weighted homogeneous germ with $n > p$, by Proposition 9.10 of [Lo]; this does not seem to imply isomorphism as E_p-modules, however.

Nonetheless, when we consider E_C and M_C as E_{p-1}- modules, then they *are* isomorphic. Since, as the proofs of (3.1) and (3.2) explain, E_C and M_C are free E_{p-1} modules, we need only see that they have the same rank. This follows from a very general (and deep) result of Buchsbaum-Rim ([BR],(4.2)) (though a considerable amount of translation of concepts is required to see this). We give an alternative proof for our particular case, since the point of view we take will also allow us to introduce two further useful notions.

(3.4) Lemma. *Let $f : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, y)$ satisfy the equivalent hypotheses of* (3.1), (3.2). *Then, in the notation of* (3.1), (3.2), $\langle \det \delta \rangle = \langle \det \Delta \rangle$ *and* $s = t$.

Proof. Let $F : (\mathbb{C}^{n+k}, S \times 0) \longrightarrow (\mathbb{C}^{p+k}, y \times 0)$ be a $(\mathbb{C}^k, 0)$-level-preserving unfolding of f which is a CS; such an F exists, since f, being of finite singularity type, has an unfolding which is infinitesimally stable, hence a CS.

We take $\widetilde{\pi} : \mathbb{C}^{p+k} \longrightarrow \mathbb{C}^{p+k-1}$ to be given by

$$(u, y_1, \ldots, y_p) \mapsto (u, y_1, \ldots, y_{p-1}),$$

and we identify the α_i of (3.1) and the β_j of (3.2) as elements of $\mathcal{E}_{S\times 0}$, resp. $\mathcal{E}^{p+k}_{S\times 0}$, in the obvious way.

The hypotheses of (3.1), (3.2) hold with f and π replaced by F and $\widetilde{\pi}$, and we obtain matrices $\widetilde{\delta}$ and $\widetilde{\Delta}$ as there. Since F is a CS, $\det \widetilde{\delta}$ and $\det \widetilde{\Delta}$ give reduced equations for $D(F)$, and so $\det \widetilde{\delta} = \det \widetilde{\Delta}$.

The δ, Δ of (3.1), (3.2) for f, π are obtained from those for F, $\widetilde{\pi}$ by restricting to $\{u = 0\}$, since restricting an equation

$$F_p \alpha_j = \sum \widetilde{v}_{ij} \alpha_i, \quad \widetilde{v}_{ij} \in E_{p+k-1}$$

to $\{u = 0\}$ gives an equation

$$f_p \alpha_j = \sum v_{ij} \alpha_i, \text{ with } v_{ij}(y) = \widetilde{v}_{ij}(0, y),$$

so the columns of $\widetilde{\delta}$ restrict to those of δ, and similarly for Δ.

Thus

$$\det \delta = \det \widetilde{\delta} | \{u = 0\}$$

so

$$\det \delta = \det \Delta,$$

proving the first statement.

Now for the second statement. Recall that $\delta = y_p I - V$, where V is a matrix with entries in $\pi^* \mathcal{E}_{p-1,\pi(y)}$. Thus V is the matrix corresponding to multiplication by f_p in the free E_{p-1}-module E_C. Restricting to $K_\pi = \pi^{-1}\pi(y)$, we see that $V|K_\pi$ is the matrix corresponding to multiplication by f_p in the $\mathbb{C}$-vector space $E_C/(\pi \circ f)^* m_{\pi(y)} E_C$. Since $\dim_{\mathbb{C}} E_C/(\pi \circ f)^* m_{\pi(y)} E_C$ is finite,

$$f_p^k \in m_p^k \subset J(f) + (\pi \circ f)^* m_{\pi(y)} E_S$$

for some finite k; thus $V|K_\pi$ is nilpotent, and thus its characteristic polynomial is $y_p^s|K_\pi$. But this characteristic polynomial is $\det(y_p I - V)|K_\pi = \det \delta|K_\pi$. So $\det \delta$ is a monic polynomial in y_p with coefficients in $\pi^* m_{\pi(y)}$ and leading term y_p^s.

In an exactly similar way, $\det \Delta$ is a monic polynomial in y_p with coefficients in $\pi^* m_{\pi(y)}$ and leading term y_p^t. So, since $\langle \delta \rangle = \langle \Delta \rangle$, we see that $s = t$, as required. □

Notice that the rest of the proof of (3.2) is now immediate.

Another consequence of this is the following:

(3.5) Corollary. *If f is a CS-FST, $s = t =$ the intersection number of K_π and $D(f)$.*

Proof. We saw during the proof that the restriction of $\det(\delta)$ to the line K_π is of order s. If f is a CS, then $\det(\delta)$ is the defining equation of $D(f)$. □

There is yet another number equal to these.

(3.6) Proposition ([Lê], [Gr]). *$s = t = \mu(f) + \mu(\pi \circ f)$.*

(Here μ denotes the Milnor number). (An important special case of this is an immediate consequence of the proof of (3.4): if f is a function (i.e., $p = 1$), then $\det \delta = < y^\mu > E_1$.)

We can use this to give a useful way of finding a basis for $M_C/(\pi \circ f)^* m_{\pi(y)} M_C$, a result which was worked out jointly with C. T. C. Wall.

(3.7) Proposition. *Suppose $n \geq p$, let $f : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, y)$ be a $\mathbb{C}$-analytic map-germ, and suppose that*

$$(0, f_p) \subset df(\mathcal{E}_S^n) + (\pi \circ f)^* m_{\pi(y)} \mathcal{E}_S^p.$$

Let $\{ \phi_i : i = 1, \dots, a \}$ be a $\mathbb{C}$-basis for

$$\mathcal{E}_S^{p-1}/d(\pi \circ f)(\mathcal{E}_S^n) + (\pi \circ f)^* m_{\pi(y)} \mathcal{E}_S^{p-1},$$

and let $\{ \psi_j : j = 1, \dots, b \}$ be a $\mathbb{C}$-basis for

$$\mathcal{E}_S/J(f) + f^* m_y \mathcal{E}_S.$$

Then

$$\{ (\phi_i, 0) : i = 1, \dots, a \} \cup \{ (0, \psi_j) : j = 1, \dots, b \}$$

is a $\mathbb{C}$*-basis for* $\mathcal{E}_S^p/df(\mathcal{E}_S^n)+(\pi\circ f)^* m_{\pi(y)}\mathcal{E}_S^p$.

Proof. “Forgetting” the final coordinate induces a surjection

$$\frac{\mathcal{E}_S^p}{df(\mathcal{E}_S^n)+(\pi\circ f)^* m_{\pi(y)}\mathcal{E}_S^p} \longrightarrow \frac{\mathcal{E}_S^{p-1}}{d(\pi\circ f)(\mathcal{E}_S^n)+(\pi\circ f)^* m_{\pi(y)}\mathcal{E}_S^{p-1}}.$$

Its kernel is isomorphic to a quotient of $\mathcal{E}_S$ by an ideal containing $J(f)+(\pi\circ f)^* m_{\pi(y)}\mathcal{E}_S$, since this last term is in the annihilator of the left hand side. Indeed, the property of $(0,f_p)$ in the hypotheses shows that the ideal also contains f_p, and so contains $J(f)+f^* m_y\mathcal{E}_S$.

If we set

$$\begin{aligned}\tau(\pi\circ f)&=\dim_{\mathbb{C}}\{\mathcal{E}_S^{p-1}/d(\pi\circ f)(\mathcal{E}_S^n)+(\pi\circ f)^* m_{\pi(y)}\mathcal{E}_S^{p-1}\},\\ \tau'(f)&=\dim_{\mathbb{C}}\{\mathcal{E}_S/J(f)+f^* m_y\mathcal{E}_S\},\end{aligned}$$

which is a slightly adapted version of the notation of [Lo], then we have shown, by (3.6), that

$$\mu(f)+\mu(\pi\circ f)\leq\tau'(f)+\tau(\pi\circ f).$$

Now

$$\tau'(f)\leq\mu(f)$$

(for $n-p>0$, this is proved on page 164 of [Lo]; if $n=p$, we always have $\mu(f)=\tau'(f)$, as shown for example in [Wa]) and

$$\tau(\pi\circ f)\leq\mu(\pi\circ f)$$

(as is shown in [LS]). So in fact

$$\tau'(f)=\mu(f) \text{ and } \tau(\pi\circ f)=\mu(\pi\circ f),$$

and our spanning set is a basis. □

We have in mind in particular the case where f is a weighted homogeneous map-germ. If, for each point in S, f is weighted homogeneous, then we have

$$(d_1 f_1,\ldots,d_p f_p)=\sum_{i=1}^n w_i\frac{\partial f}{\partial x_i},$$

where w_i are the weights allocated to coordinates x_i, and d_i are the weights of the f_i.

(3.8). The foregoing results provide a discriminant matrix bigger than actually necessary if $s>k=\dim_{\mathbb{C}}\{E_C/f^* m_y E_C\}$ or if $t>k'=\dim_{\mathbb{C}}\{M_C/f^* m_y M_C\}$: there must be a $k\times k$ discriminant matrix of E_C and a $k'\times k'$ discriminant matrix of M_C. In fact, we can find these matrices and the presentations they represent from those of (3.1) and (3.2), if we choose the α_i and β_j in the right way.

Since $\dim_{\mathbb{C}}\{E_C/(\pi\circ f)^* m_{\pi(y)} E_C\} < \infty$, there is some $r < \infty$ such that

$$f_p^r \mathcal{E}_S \subset J(f) + (f_1, \ldots, f_{p-1})\mathcal{E}_S,$$

and thus, if $a_1, \ldots, a_k$ project to a basis for $E_C/f^* m_y E_C$, then

$$\{\, f_p^l a_j : j = 1, \ldots, k,\ l = 0, \ldots, r-1 \,\}$$

project to a spanning set for $E_C/(\pi\circ f)^* m_{\pi(y)} E_C$. Thus we can choose a collection of elements

$$\{\, \alpha_{lj} = f_p^l a_j : j = 1, \ldots, k,\ l = 0, \ldots, r_j \,\} \tag{3.8.1}$$

which project to a basis.

We then have $r - k$ "trivial" relations

$$\alpha_{lj} = f_p \alpha_{l-1,j}, \quad j = 1, \ldots, k,\ l = 1, \ldots, r_j; \tag{3.8.2}$$

substituting $f_p^l a_j$ for α_{lj} in the remaining k relations gives a $k \times k$-discriminant matrix derived from E_C. We illustrate this procedure in Example (3.9.6) below.

An exactly similar argument gives a $k' \times k'$-discriminant matrix derived from M_C.

The matrices so produced are canonical in the sense that a different choice of $\mathcal{E}_y$-basis for the module E_C or M_C, and a different choice of basis for the module of relations induces an $\mathcal{E}_y$-equivalence of matrices.

(3.9) Examples.

1) Let $f(x) = (x^2) = (Y)$ (i. e., Y is the target coordinate). The above algorithm produces, with respect to the E_p- (here $p = 1$) and E_{p-1}-basis $\{[1]\}$ in E_C, the discriminant matrix:

$$\delta = [\, Y \,].$$

2) Let $f(u, x) = (u, x^3 + ux) = (U, Y)$. The above algorithm produces, with respect to the E_p- and E_{p-1}-basis $\{[1], [x]\}$ in E_C, the discriminant matrix:

$$\delta = \begin{bmatrix} Y & \frac{2}{9}U^2 \\ -\frac{2}{3}U & Y \end{bmatrix}.$$

3) Let $f(u, v, x) = (u, v, x^4 + ux + vx^2) = (U, V, Y)$. The above algorithm produces, with respect to the E_p- and E_{p-1}-basis $\{[1], [x], [x^2]\}$ in E_C, the discriminant matrix:

$$\delta = \begin{bmatrix} Y & \frac{1}{8}UV & \frac{3}{16}U^2 \\ -\frac{3}{4}U & Y + \frac{1}{4}V^2 & \frac{1}{2}UV \\ -\frac{1}{2}V & -\frac{3}{4}U & Y + \frac{1}{4}V^2 \end{bmatrix}.$$

4) Let $f(u, v, w, x) = (u, v, w, x^5 + ux + vx^2 + wx^3) = (U, V, W, Y)$. The above algorithm produces, with respect to the E_p- and E_{p-1}-basis $\{[1], [x], [x^2], [x^3]\}$ in E_C, the discriminant matrix:

$$\delta = \begin{bmatrix} Y & \frac{2}{25}UW & \frac{3}{25}UV & \frac{4}{25}U^2 - \frac{6}{125}UW^2 \\ -\frac{4}{5}U & Y + \frac{4}{25}VW & \frac{2}{25}UW + \frac{6}{25}V^2 & \frac{11}{25}UV - \frac{12}{25}VW^2 \\ -\frac{3}{5}V & -\frac{4}{5}U + \frac{6}{25}W^2 & Y + \frac{13}{25}VW & \frac{14}{25}UW + \frac{6}{25}V^2 - \frac{18}{125}W^3 \\ -\frac{2}{5}W & -\frac{3}{5}V & -\frac{4}{5}U + \frac{6}{25}W^2 & Y + \frac{13}{25}VW \end{bmatrix}.$$

Discriminant matrices, though with respect to different bases, are produced for the stable unfoldings of *all* x^k in [Ar2], though they are obtained from a rather different point of view. We will explain the connection a little later on (Proposition (3.12)). Also the determinants (up to a unit) are calculated in [Te].

5) Let $f(u,v,w,x,y) = (u,v,w,x^3+y^3+ux+vy+wxy) = (U,V,W,Z)$. The above algorithm produces, with respect to the E_p- and E_{p-1}-basis $\{[1],[x],[y],[xy]\}$ in E_C, the discriminant matrix:

$$\delta = \begin{bmatrix} Z & \frac{2}{9}U^2 - \frac{1}{27}VW^2 & \frac{2}{9}V^2 - \frac{1}{27}UW^2 & -\frac{5}{27}UVW \\ -\frac{2}{3}U & Z - \frac{1}{27}W^3 & \frac{1}{3}VW & \frac{2}{9}V^2 - \frac{1}{9}UW^2 \\ -\frac{2}{3}V & \frac{1}{3}UW & Z - \frac{1}{27}W^3 & \frac{2}{9}U^2 - \frac{1}{9}VW^2 \\ -\frac{1}{3}W & -\frac{2}{3}V & -\frac{2}{3}U & Z - \frac{1}{27}W^3 \end{bmatrix}.$$

Observe that the discriminant matrices above are also those for stable unfoldings of functions obtained by adding a nondegenerate quadratic form in extra variables, since E_C is unaffected.

6) Let $f(u,v,x,y) = (u,v,xy,x^2+y^2+ux+vy) = (U,V,W,Z)$. The above algorithm produces, with respect to the E_{p-1}-basis $\{[1],[x],[y],[Z\circ f]\}$ in E_C, the discriminant matrix:

$$\delta_1 = \begin{bmatrix} Z & -\frac{3}{2}VW & -\frac{3}{2}UW & -4W^2 - \frac{7}{2}UVW \\ 0 & Z + \frac{3}{8}U^2 & -2W + \frac{1}{8}UV & -4VW + \frac{3}{16}U^3 + \frac{1}{16}UV^2 \\ 0 & -2W + \frac{1}{8}UV & Z + \frac{3}{8}V^2 & -4UW + \frac{3}{16}V^3 + \frac{1}{16}U^2V \\ -1 & -\frac{1}{4}U & -\frac{1}{4}V & Z - \frac{1}{8}U^2 - \frac{1}{8}V^2 \end{bmatrix}.$$

An E_p-basis for E_C is $\{[1],[x],[y]\}$; we chose $[Z\circ f]$ as the fourth generator as an E_{p-1}-basis to satisfy (3.8.1). The first column represents the "trivial relation" of (3.8.2). We reduce the number of generators and relations by 1 by making the substitution $Z\cdot[1]$ for $[Z\circ f]$; in the language of matrix theory, we do the elementary row operation: replace the first row by the first row plus Z times the fourth row—then remove the first column and fourth row.

With respect to this E_p-basis, this procedure yields the discriminant matrix:

$$\delta_2 = \begin{bmatrix} -\frac{3}{2}VW - \frac{1}{4}UZ & -\frac{3}{2}UW - \frac{1}{4}VZ & -4W^2 - \frac{7}{2}UVW + Z^2 - \frac{1}{8}(U^2+V^2)Z \\ Z + \frac{3}{8}U^2 & -2W + \frac{1}{8}UV & -4VW + \frac{3}{16}U^3 + \frac{1}{16}UV^2 \\ -2W + \frac{1}{8}UV & Z + \frac{3}{8}V^2 & -4UW + \frac{3}{16}V^3 + \frac{1}{16}U^2V \end{bmatrix}.$$

Now let us compute a discriminant matrix using M_C. $\{[(1,0)],[(0,1)],[(0,x)],[(0,y)]\}$ is an E_p- and an E_{p-1}-basis for M_C. The algorithm of (3.2) produces the discriminant matrix:

$$\Delta = \begin{bmatrix} Z & W & \frac{1}{4}UW & \frac{1}{4}VW \\ 4W + 2UV & Z & -\frac{3}{2}VW & -\frac{3}{2}UW \\ 3V & -\frac{1}{2}U & Z + \frac{1}{4}U^2 & -2W \\ 3U & -\frac{1}{2}V & -2W & Z + \frac{1}{4}V^2 \end{bmatrix}.$$

It is interesting to note how much simpler Δ is than either δ_1 or δ_2.

The reader is invited to calculate the equations of the respective discriminants by computing determinants.

The examples we have given are all of stable germs; this is because discriminant matrices for non-stable germs (of FST) can be induced from these, as we will now show.

We have concentrated on the case $n \geq p$ so far in this section; *everything we do from now on will be for general n and p.*

Recall that a *morphism* of map-germs $f \mapsto g$ is a pair of map-germs (ϕ, ψ) such that the diagram

$$\begin{array}{ccc} (N', S') & \xrightarrow{g} & (P', y_0') \\ \uparrow \phi & & \uparrow \psi \\ (N, S) & \xrightarrow{f} & (P, y_0) \end{array}$$

is cartesian; that is, it commutes, ψ is transverse to g, and (ϕ, f) is a diffeomorphism of (N, S) onto the fibre product of g and ψ.

An important particular case of a morphism occurs when ϕ, ψ are embeddings, in which case $(g; (\phi, \psi))$ is called an *unfolding* of f. It is well-known that any map-germ of FST has a stable unfolding: that is, has an unfolding $(g; (\phi, \psi))$ with g stable.

At the opposite extreme, we have the case where ϕ, ψ are submersions: we call this a *projection.*

In point of fact, any morphism can be written as the composite of an unfolding and a projection; for (ϕ, ψ) as above is the composite of the unfolding

$$((\phi, f), (\psi, 1_P)) : f \mapsto g \times 1_P$$

and the projection

$$(\pi_P, \pi'_P) : g \times 1_P \mapsto g$$

where $\pi_P : (N' \times P, S \times y_0) \longrightarrow (N', y_0)$ and $\pi'_P : (P' \times P, y_0' \times y_0) \longrightarrow (P', y_0')$ are the natural projections. In fact a suitable choice of coordinates allows any projection to be written in the above form. Similarly, a suitable choice of coordinates allows any unfolding to be written in the form

$$\begin{array}{ccc} (N \times U, S \times 0) & \xrightarrow{g} & (P \times U, y_0 \times 0) \\ \uparrow 1_N \times 0 & & \uparrow 1_P \times 0 \\ (N, S) & \xrightarrow{f} & (P, y_0) \end{array}$$

where U is an open neighborhood of 0 in some Euclidean space, and g is $(U, 0)$-level-preserving (see e.g. the argument after Definition (0.1), Chapter III, of [GWPL]).

One basic fact is that Jacobian ideals "commute with base change" (see page 48 of [Lo]): $\phi^* J(g) \cdot \mathcal{E}_S = J(f)$. This is immediate if the morphism is a projection, and also if the morphism is an unfolding, so holds for all morphisms. Also the discriminant space commutes with base change (see page 62 of [Lo] and also [Te]): $F_0(E_{D(f)})$ is generated by $F_0(E_{D(g)}) \circ \psi$. We show that similar properties hold for the discriminant matrices.

(3.10) Theorem. *Suppose that $(\phi, \psi) : f \mapsto g$ is an analytic morphism of map-germs as above, with N, N', P and P' replaced by $\mathbb{C}^n$, $\mathbb{C}^{n'}$, $\mathbb{C}^p$ and $\mathbb{C}^{p'}$. Let $\beta_1, \ldots, \beta_k \in \mathcal{E}_{S'}$ project to a basis for $E_{C'}$ as an $E_{p'}$-module, and let*

$$E^l_{p'} \xrightarrow{\delta'} E^k_{p'} \xrightarrow{\pi'} E_{C'} \to 0$$

be a presentation with $\pi'(e_i) = \beta_i$.

Let $\alpha_1, \ldots, \alpha_k \in \mathcal{E}_S$ be such that

$$\phi^* \beta_i - \alpha_i \in J(f) \qquad (i = 1, \ldots, k).$$

Then

$$E^l_p \xrightarrow{\delta} E^k_p \xrightarrow{\pi} E_C \to 0,$$

with $\delta_{ij} = \delta'_{ij} \circ \psi$ and $\pi(e_i) = \alpha_i$, is a presentation for E_C (and δ is injective if δ' is).

We omit the proof, which is very similar to the proof of the following theorem.

(3.11) Theorem. *Suppose that $(\phi, \psi) : f \mapsto g$ is an analytic morphism of map-germs as above, with N, N', P and P' replaced by $\mathbb{C}^n$, $\mathbb{C}^{n'}$, $\mathbb{C}^p$ and $\mathbb{C}^{p'}$. Let $\beta_1, \ldots, \beta_k \in \mathcal{E}^{p'}_{S'}$ project to a basis for $M_{C'}$ as an $E_{p'}$-module, and let*

$$E^l_{p'} \xrightarrow{\Delta'} E^k_{p'} \xrightarrow{\pi'} M_{C'} \to 0$$

be a presentation with $\pi'(e_i) = \beta_i$.

Let $\alpha_1, \ldots, \alpha_k \in \mathcal{E}^p_S$ be such that

$$\phi^* \beta_i - d\psi \circ \alpha_i \in dg(\mathcal{E}^{n'}_{S'}) \circ \phi \qquad (i = 1, \ldots, k).$$

Then

$$E^l_p \xrightarrow{\Delta} E^k_p \xrightarrow{\pi} M_C \to 0,$$

with $\Delta_{ij} = \Delta'_{ij} \circ \psi$ and $\pi(e_i) = \alpha_i$, is a presentation for M_C (and Δ is injective in case Δ' is).

Proof. If $(R_1, \ldots, R_k) \in E^k_{p'}$ is a relation for $\beta_1, \ldots, \beta_k$ modulo $dg(\mathcal{E}^{n'}_{S'})$, then we have

$$\sum_{i=1}^{k} g^* R_i \cdot \beta_i = dg \cdot \eta, \text{ for some } \eta \in \mathcal{E}^{n'}_{S'},$$

so, composing with ϕ, we have

$$\sum_{i=1}^{k} (g \circ \phi)^* R_i \cdot \beta_i \circ \phi = dg \cdot \eta \circ \phi.$$

Since $g \circ \phi = \psi \circ f$ and $\beta_i \circ \phi - d\psi \cdot \alpha_i \in dg(\mathcal{E}_{S'}^{n'}) \circ \phi$, this gives

$$\sum_{i=1}^{k} (\psi \circ f)^* R_i \cdot d\psi \cdot \alpha_i = dg \cdot \eta' \circ \psi, \text{ for some } \eta' \in \mathcal{E}_{S'}^{n'},$$

or

$$d\psi \cdot (\sum_{i=1}^{k} (\psi \circ f)^* R_i \cdot \alpha_i) = dg \cdot \eta' \circ \phi.$$

Thus $(\sum_{i=1}^{k} (\psi \circ f)^* R_i \cdot \alpha_i, \eta' \circ \phi)$ is tangent to the fibre-product of ψ and g; since (f, ϕ) is a diffeomorphism to this fibre product, there is a vector field $\zeta \in \mathcal{E}_S^n$ such that

$$df \cdot \xi = \sum_{i=1}^{k} (\psi \circ f)^* R_i \cdot \alpha_i$$

and

$$d\phi \cdot \zeta = \eta' \circ \phi;$$

in particular, $(\psi^* R_1, \ldots, \psi^* R_k)$ is a relation for $\alpha_1, \ldots, \alpha_k$ modulo $df(\mathcal{E}_S^n)$.

It follows that the columns of Δ are relations for the α_i modulo $df(\mathcal{E}_S^n)$; so it remains to be shown that any relation for $\alpha_1, \ldots, \alpha_k$ modulo $df(\mathcal{E}_S^n)$ can be obtained as an E_p-linear combination of these columns. As in (3.10), we need only treat the cases of unfolding and projection.

Consider first the case where (ϕ, ψ) is an unfolding. There is an l such that $n' = n + l$ and $p' = p + l$. Via coordinate choice we can treat g as a $(\mathbb{C}^l, 0)$-level-preserving map-germ with 0-level f, with $\phi = 1_N \times \{0\}$, $\psi = 1_P \times \{0\}$; we write these as i, j respectively. Let $(S_1, \ldots, S_k)$ be a relation for $\alpha_1, \ldots, \alpha_k$ modulo $df(\mathcal{E}_S^n)$, so

$$\sum_{i=1}^{k} f^* S_i \cdot \alpha_i \in df(\mathcal{E}_S^n),$$

whence

$$dj(\sum_{i=1}^{k} f^* S_i \cdot \alpha_i) \in dj \circ df(\mathcal{E}_S^n) \subset dg \circ di(\mathcal{E}_S^n).$$

Hence, if the $\widetilde{S}_i$ are extensions of the S_i on $(\mathbb{C}^p \times \mathbb{C}^l, 0)$, then, since $dj \cdot \alpha_i - \beta_i \circ i \in dg(\mathcal{E}_{S'}^{n'}) \circ i$,

$$(\sum_{i=1}^{k} g^* \widetilde{S}_i \cdot \beta_i) \circ i \in dg(\mathcal{E}_{S'}^{n'}) \circ i,$$

so that

$$\sum_{i=1}^{k} g^* \widetilde{S}_i \cdot \beta_i \in dg(\mathcal{E}_{S'}^{n'}) + m_l \mathcal{E}_{S'}^{p'}.$$

Since $\mathcal{E}_{S'}^{p'} = dg(\mathcal{E}_{S'}^{n'}) + \{\beta_1, \ldots, \beta_k\} g^* E_{p'}$, we thus have

$$\sum_{i=1}^{k} g^* \widetilde{S}_i \cdot \beta_i - \sum_{i=1}^{k} g^* T_i \cdot \beta_i \in dg(\mathcal{E}_{S'}^{n'})$$

for some $T_i \in m_l E_{p'}$, or

$$\sum_{i=1}^{k} g^*(\widetilde{S}_i - T_i) \cdot \beta_i \in dg(\mathcal{E}_{S'}^{n'}).$$

Thus $(\widetilde{S}_1 - T_1, \ldots, \widetilde{S}_k - T_k)$ is an $E_{p'}$-linear combination of the columns of Δ'; composing with j, we see $(S_1, \ldots, S_k)$ as an E_p-linear combination of the columns of Δ, as required.

Now we treat the case where (ϕ, ψ) is a projection. By suitable coordinate choice we can suppose $f = g \times 1_{\mathbb{C}^l}$, with ϕ the projection of $\mathbb{C}^{n'} \times \mathbb{C}^l$ to $\mathbb{C}^{n'}$ and ψ the projection of $\mathbb{C}^{p'} \times \mathbb{C}^l$ to $\mathbb{C}^{p'}$.

Let $R_f = \mathrm{im}(\Delta)$ and $R_g = \mathrm{im}(\Delta')$ be the modules of relations. Choose $S = (S_1, \ldots, S_k)$, an element of R_f. In local coordinates, we can write

$$\begin{aligned} f(x,u) &= (g(x), u), \\ \alpha_i(x,u) &= (\beta_i(x), \gamma_i(x,u)) \qquad \text{for some } \gamma_i, \\ S_i(x,u) &= S_i^0(x) + \sum_{j=1}^{l} u_j S'_{i,j}(x,u). \end{aligned}$$

Then $S^0 = (S_1^0(x), \ldots, S_k^0(x)) \in R_g$. Furthermore,

$$\sum_{i=1}^{k} S_i(g(x), u)\beta_i(x) = dg\eta(x,u), \qquad \text{for some } \eta.$$

Thus,

$$\sum_{i=1}^{k} (S_i(g(x), u) - S_i(g(x), 0))\beta_i(x) = dg(\eta(x,u) - \eta(x,0)) \in m_l dg(\mathcal{E}_{S'}^{n'}).$$

It follows that $S - S^0 \in m_l R_f$.

Thus

$$R_f \subset (\psi)^* E_{p'} R_g + m_l \cdot R_f,$$

hence

$$\begin{aligned} R_f &\subset ((\psi)^* E_{p'} R_g) E_{p'+l} \\ &= ((\psi)^* E_{p'} \langle \text{columns of } \Delta' \rangle) E_{p'+l} \\ &= \langle \text{columns of } \Delta' \rangle E_p, \end{aligned}$$

as required.

□

The discriminant matrix Δ of an M_C is not unique of course. In particular, it depends on a choice of generating set in M_C. If F is a stable mapping, then any basis for the constant vector fields in the target projects to a generating set for M_C as a module over the target. (Important note: it is not necessarily a minimal generating set; the target dimension minus the cardinality of a minimal generating set is the dimension of the $\mathcal{A}$-equisingularity set in the target.) This leads to a very interesting and useful discriminant matrix, which has been studied from various points of view, and in varying generality, in [Ar2], [Te], [Sa] and [Lo].

Let $F : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, 0)$ be a map-germ of FST with discriminant D. Let $\Theta(D)$ denote the vector fields on $(\mathbb{C}^p, 0)$ liftable over F, i.e. those vector fields η on $(\mathbb{C}^p, 0)$ for which there exists a vector field ξ on $(\mathbb{C}^n, S)$ with $dF_x(\xi_x) = \eta(f(x))$ for all $x \in (\mathbb{C}^n, S)$. If F is FST and a CS, this module of vector fields coincides exactly with the module of vector fields tangent to the discriminant of F. For the $(\mathbb{C}, 0)$-level-preserving diffeomorphism of $(\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$ induced by integrating such a vector field η preserves the discriminant of $F \times 1_{\mathbb{C}}$, so by [duP2] lifts over $F \times 1_{\mathbb{C}}$ to a (necessarily $\mathbb{C}$-level-preserving) diffeomorphism of $(\mathbb{C}^n \times \mathbb{C}, S \times 0)$. Differentiating this in the $\mathbb{C}$- direction yields a vector field on $(\mathbb{C}^n, S)$ lifting η. Conversely, integrating a liftable vector field η' and its lift yields a self-$\mathcal{A}$-equivalence of $F \times 1_{\mathbb{C}}$, whose target diffeomorphism thus preserves the discriminant. Differentiating this in the $\mathbb{C}$-direction yields η' as a vector field preserving the discriminant.

(3.12) Proposition. *Let $F : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, 0)$ be an infinitesimally stable map-germ, so that $\frac{\partial}{\partial y_i} \circ F$ $(i = 1, \ldots, p)$ project to an F^*E_p-generating set for M_C. Choose a resolution*

$$E_p^q \xrightarrow{\Delta} E_p^p \xrightarrow{\pi} M_C \to 0$$

such that $\pi(e_i) = [\frac{\partial}{\partial y_i} \circ F]$ $(i = 1, \ldots, p)$. Then the vector fields $\eta_j = \sum_{i=1}^p \Delta_{ij} \frac{\partial}{\partial y_i}$ $(j = 1, \ldots, q)$(i.e., the columns of Δ) generate $\Theta(D)$ as an E_p-module.

Proof. The E_p-linear map

$$E_p^p \longrightarrow \Gamma(T(\mathbb{C}^p, 0))$$

given by $e_i \mapsto \frac{\partial}{\partial y_i}$ is an isomorphism of free E_p-modules; thus all that we need to do is to see that it carries the relation module of the $\pi(e_i)$ onto the module of vector fields liftable over F.

We have $(A_1, \ldots, A_p)$ is a relation for $\frac{\partial}{\partial y_1} \circ F, \ldots, \frac{\partial}{\partial y_p} \circ F$ modulo $\mathrm{im}(dF)$ iff

$$\sum_{i=1}^p A_i \circ F \cdot \frac{\partial}{\partial y_i} \circ F \in \mathrm{im}(dF)$$

iff

$$\sum_{i=1}^{p} A_i \cdot \frac{\partial}{\partial y_i} \text{ is a liftable vector field,}$$

as required. □

(3.13) Corollary. *Let $F : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, 0)$ be infinitesimally stable, with $n \geq p$. The E_p-module of vector fields on $(\mathbb{C}^p, 0)$ tangent to the discriminant of F is free, of rank p.*

As a demonstration of the power of Theorem 3.10 and Proposition 3.11, we give a simple proof of a generalization of a theorem of Damon ([Da]).

Let $f : (\mathbb{C}^n, S) \longrightarrow (\mathbb{C}^p, 0)$ be a map-germ of FST. Then we can choose elements of $\mathcal{E}_S^p$, projecting to a generating set of $M_C = \mathcal{E}_S^p / df(\mathcal{E}_S^n)$, of the form

$$\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p}, \alpha_1, \ldots, \alpha_s,$$

with the $\alpha_i \in m_S \mathcal{E}_S^p$. Then

$$F(u, x) = (u, f(x) + \sum_{i=1}^{s} u_i \alpha_i)$$

is an infinitesimally stable unfolding of f. Let ϕ and ψ be the inclusions of $\mathbb{C}^n$ and $\mathbb{C}^p$ into $\mathbb{C}^{n+s}$ snd $\mathbb{C}^{p+s}$, respectively. Let C and D denote the critical and discriminant sets of f and C' and D' those of F. Let $p' = p + s$ and $n' = n + s$. Then $\beta_1, \ldots, \beta_{p'} =$

$$\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_s}$$

project to a generating set of $M_{C'}$ as an E_{p+s}-module, and are related to $\gamma_1, \ldots, \gamma_{p'} =$

$$\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p}, -\alpha_1, \ldots, -\alpha_s,$$

as in Theorem (3.11), i. e., $\phi^* \beta_i - d\psi \circ \gamma_i \in dF(E_{n'}^{p'}) \circ \phi$ for $i = 1, \ldots, p'$. Let

$$E_{p'}^l \xrightarrow{\Delta'} E_{p'}^{p'} \xrightarrow{\pi'} M_{C'} \longrightarrow 0$$

be a presentation with $\pi'(e_i) = \beta_i$. Then

$$E_p^l \xrightarrow{\Delta} E_p^{p'} \xrightarrow{\pi} M_C \longrightarrow 0,$$

with $\Delta_{ij} = \Delta'_{ij} \circ \psi$ and $\pi(e_i) = \gamma_i$, is a presentation for M_C. Thus

$$N\mathcal{R}_e \cdot f = M_C = \frac{\mathcal{E}_S^p}{df(\mathcal{E}_S^n)} \cong \frac{E_p^{p'}}{\mathrm{im}(\Delta' \circ \psi)} = N\mathcal{C}_{D',e} \cdot \psi$$

as an E_p-module, and $\mathrm{im}(\Delta' \circ \psi) = \Theta(D') \circ \psi$. This isomorphism then induces the isomorphism of:

(3.14) Theorem. *With notation as above, we have an isomorphism of E_p-modules*

$$N\mathcal{A}_e \cdot f = \frac{\mathcal{E}^p_S}{df(\mathcal{E}^n_S) + f^* E^p_p} \cong \frac{E^{p+s}_p}{\Theta(D') \circ \psi + \langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p} \rangle \cdot E_p} = N\mathcal{K}_{D',e} \cdot \psi.$$

The notation $N\mathcal{K}_{D',e} \cdot \psi$ is Damon's for the normal space to the action of the extended $\mathcal{K}_{D'}$-equivalence, see [Da] ($\mathcal{C}_{D'}$ is the subgroup of $\mathcal{K}_{D'}$ of elements leaving the domain of ψ fixed). Damon proved this under the extra assumption that f was of finite $\mathcal{A}$-codimension. We only assume FST. In the finite $\mathcal{A}$-codimension case, it follows immediately that the $\mathcal{A}_e$-codimension of f equals the $\mathcal{K}_{D',e}$-codimension of ψ.

As another application, we look at the computation of the *stability discriminant*, which is the set of points in the target at which a mapping is not infinitesimally stable. This plays an important role in the study of topological equisingularity.

Continuing with the above notation, f is stable iff

$$\Theta(D') \circ \psi + \langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_p} \rangle \cdot E_p \tag{3.15}$$

is all of E^{p+s}_p. But (3.15) is the image of the matrix $\widetilde{\Delta}$ formed by augmenting Δ with columns forming a basis for $\mathbb{C}^p$. Then the Jacobian ideal of $\widetilde{\Delta}$, which is the ideal generated by the $s \times s$-determinants of the first s-rows of Δ, defines the stability discriminant.

(3.16) Example. Let $f(x, y) = (x, y^3 + Q(x)y)$, with $x = (x_1, \ldots, x_{n-2})$. A stable unfolding of f is $F(u, x, y) = (u, x, y^3 + uy + Q(x)y)$. Let (U, X, Y) be the target coordinates. Let $g = (g_1, \ldots, g_{n-2})$, where $g_i = \partial Q/\partial x_i$, and $h = u + Q(x)$; note that both these functions lift to functions of U and X: $g(X)$ and $h(U, X)$. Then

$$dF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & g \cdot y & 3y^2 + h \end{bmatrix}.$$

We calculate a discriminant matrix for $M_{C'}$ relative to the generating set

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

$F_n = y^3 + hy$. First observe that, modulo $dF(E^n_n)$, $ye_n \equiv -e_1$, $y^2 e_n \equiv -\frac{1}{3}he_n$, $y^3 e_n \equiv -\frac{1}{3}hye_n \equiv \frac{1}{3}he_1$ and $y^4 e_n \equiv \frac{1}{9}h^2 e_n$. Thus

$$F_n \cdot e_1 \equiv -(y^4 + hy^2)e_n \equiv \tfrac{2}{9}h^2 e_n,$$
$$F_n \cdot e_n \equiv -\tfrac{2}{3}e_1.$$

Also we have the trivial relations

$$e_i = -g_{i-1}ye_n \equiv g_{i-1}e_1, \qquad 1 < i < n.$$

Thus we have

$$\Delta' = \begin{bmatrix} Y & -g & \frac{2}{3}h \\ 0 & 1 & 0 \\ -\frac{2}{9}h^2 & 0 & Y \end{bmatrix}.$$

So the columns of Δ' form a free E_n basis of $\Theta(D')$. (A minimal generating set for $M_{C'}$ as an F^*E_n-module would be for example e_1 and e_n, reflecting that there is a nonsingular $n-2$-dimensional surface of points in the target along which F is $\mathcal{A}$-equisingular; the middle $n-2$ columns are tangent to this surface). D' is defined by $\det \Delta' = Y^2 + \frac{4}{27}h^3 = 0$ and D is defined by the restriction of this to $u = 0$, namely $Y^2 + \frac{4}{27}Q^3 = 0$. We have a representation

$$E_{n-1}^{n+2} \xrightarrow{\widetilde{\Delta}} E_{n-1}^{n} \longrightarrow \frac{E_{n-1}^{n-1}}{df(E_{n-1}^{n-1}) + f^*(E_{n-1}^{n-1})} \longrightarrow 0$$

where

$$\widetilde{\Delta} = \begin{bmatrix} Y & -g & \frac{2}{3}Q & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -\frac{2}{9}Q^2 & 0 & Y & 0 & 1 \end{bmatrix}.$$

Thus the stability discriminant is the ideal generated by the $n \times n$-determinants of this matrix, i.e. $\langle Y, Q, g_1, \ldots, g_{n-2}\rangle E_{n-1}$; the locus is just the locus of $\mathcal{K}$-instability of Q, so f is finitely $\mathcal{A}$-determined if, and only if, Q is finitely $\mathcal{K}$-determined. Its $\mathcal{A}_e$-codimension equals the $\mathbb{C}$ dimension of the cokernel of $\widetilde{\Delta}$, which is the $\mathcal{K}_e$-codimension of Q.

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