Some remarks about ∞-determinacy of germs with line singularities

Hans Brodersen

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Proof of (1) $\Rightarrow$ (4) (of Theorem 4 page 2792 Sun and Wilson). The gap in the proof (of Sun and Wilson) of this implication is located in the proof of the implication that (1) implies condition (3) in the definition of isolated line singularity. We must therefore assume that condition (3) fails and construct two different realization of the given ∞-jet that cannot be $\mathcal{R}_L$-equivalent. Let $f(x,y) = \sum_{1\le i,j\le n} y_i y_j f_{i,j}(x,y)$ (with $f_{i,j} = f_{j,i}$) be the germ under consideration. We assume that condition (3) fails and consequently that $D_f(x) = \det(f_{i,j}(x,0) \in \mathbf{m}_1^\infty$. Let $\mathrm{Sym}(n)$ be the space of symmetric $n \times n$ matrices and let $\Lambda(n)$ be the subset of singular symmetric matrices. Since det is a polynomial on $\mathrm{Sym}(n)$ and $\det^{-1}(0) = \Lambda(n))$ there is a neighborhood U of $(f_{i,j}(0)) \in \mathrm{Sym}(n)$ and a constant $C > 0$ and $\alpha > 0$ such that $\det M \ge C\mathrm{dist}(M, \Lambda(n))^\alpha$ when $M \in U$. So consequently the function $\mathrm{dist}((f_{i,j}(x,0), \Lambda(n))$ must be flat in the sense $o(|x|)^m$ for every m (since the function is not necessarily differentiable)). We can therefore find a secquence $x_m \to 0$ in L and matrices $M_m = (h^m_{i,j}) \in \mathrm{Sym}(n)$ such that $\|M\|_m = o(|x|)^m$ and $(f_{i,j}(x,0)) + M_m \in \Lambda(n)$. A standard extension argument (which I do not give) shows that there exists functions $h_{i,j} \in \mathbf{m}_1^\infty$ (with $h_{i,j} = h_{j,i}$) such that $h_{i,j}(x_m) = h^m_{i,j}$. Putting

$$g = f + \sum_{1\le i,j\le n} y_i y_j h_{i,j}(x) = \sum_{1\le i,j\le n} y_i y_j g_{i,j}(x,y)$$

we get a germ with the same ∞-jet as f such that $(g_{i,j}(x,0)$ is singular along the sequence x_n. If we can construct another representative of the ∞-jet of f $k(x,y) = \sum_{1\le i,j\le n} y_i y_j k_{i,j}(x,y)$ such that $(k_{i,j}(x,0)$ is non-singular when $x \ne 0$ Lemma 4.1 in Sun and Wilson will show that g and k cannot be $\mathcal{R}_L$-equivalent hence (1) of Theorem 4 fails. So our more difficult task is the construction of such a germ k.

Consider a matrix $M \in \mathrm{Sym}(n)$. Let $1 \le r \le n$. We say that an $r \times r$-minor is an essential $r \times r$-minor of M if it is obtained by removing $n - r$ columns and lines from M corresponding to the same subset (of cardinality $n - r$) of $\{1, 2, \dots, n\}$ (so the minor is the determinant of a symmetric $r \times r$ submatrix of M). Since $D(x)$ is flat we can define the integer $0 \le s < n$ by

$$s = \max\{r \mid \text{there exists an essential } r \times r - \text{minor of } M \text{ which is not flat}\}.$$

Here $s = 0$ means that all essential $r \times r$ minors are flat for $1 \le r \le n$. For each $r > s$, let $I(r)$ be an index set with cardinality equal the number of $r \times r$ essential minors of a $n \times n$ matrix ($\sharp I(r) = \binom{n}{r}$). We index the various $r \times r$ minors of a matrix B by the set $I(r)$ and write $D_\alpha(M)$ for the corresponding $r \times r$-minor of M for each $\alpha \in I(r)$. Write $D_\alpha(f_{i,j}(x,0)) = D_\alpha(x)$ Let $l = (\sum_{r=s+1}^{n} \binom{n}{r})^2$ let $h(x)$ be a flat function such that $h(x) > 0$ when $x \neq 0$. Consider the function

$$\rho(x) = l(\sum_{r=s+1}^{n} \sum_{\alpha \in I(r)} D_\alpha(x)^2 + h(x)).$$

ρ is a flat function and $\rho(x) > 0$ when $x \neq 0$. Moreover

$$\sqrt{\rho(x)} > \sum_{r=s+1}^{n} \sum_{\alpha \in I(r)} |D_\alpha(x)|$$

when $x \neq 0$. (Because

$$(\sum_{r=s+1}^{n} \sum_{\alpha \in I(r)} D_\alpha(x)^2 + h(x))^{\frac{1}{2}} > |D_\beta(x)|$$

for each $\beta \in I(r), r > s$ and there is exactly $\sqrt{l}$ such essential minors when $s+1 \leq r \leq n$.) Assume first that $s = 0$. Let t be an integer. Since ρ is flat, we can find t flat fuctions $a_1(x), \ldots, a_t(x)$ such that $\rho(x) = a_1(x) \cdots a_t(x)$. Since $\rho(x) > 0$ when $x \neq 0$, we must have $a_i(x) \neq 0$ for each i when $x \neq 0$. Let $p(x) = \sum_{i=1}^{t} a_i(x)^2$. $p(x)$ is then flat, $p(x) > 0$ for $x \neq 0$ and if $t > 4n$ we must then have $p(x)^n > \sqrt{\rho(x)}$ when $x \neq 0$. Put $k(x,y) = f(x,y) + \sum_{i=1}^{n} y_i^2 p(x)$. Then $D_k(x) = \det((f_{i,j}(x,0) + p(x)I)$. Expanding this determinant we find that

$$D_k(x) = p(x)^n + \sum_{r=1}^{n} \sum_{\alpha \in I(r)} D_\alpha(x) p(x)^{n-r}.$$

If we are close to 0 then $|p(x)^{n-r}| < 1$ and hence

$$|\sum_{r=1}^{n} \sum_{\alpha \in I(r)} D_\alpha(x) p(x)^{n-r}| < \sqrt{\rho(x)},$$

and from estimates above, we get consequently that $D_k(x) > 0$ when $x \neq 0$. Next assume that $s > 0$. Then we can find an essential $s \times s$-minor of $(f_{i,j}(x,0)$ which is not flat. By permuting the coordinates in $\mathbf{R}^n$, we may assume that this is the the upper-left $s \times s$ minor. Denote this minor with $H(x)$. Then there exists $\beta > 0$ such that $H(x) \geq |x|^\beta$ in a neighborhood of 0. Let $\rho(x)$ be defined as above. Now $\frac{\rho(x)}{H(x)^2}$ is a flat function which is positive when $x \neq 0$ and we can for each t, write $\frac{\rho(x)}{H(x)^2} = a_1(x) \cdots a_t(x)$ for some flat functions $a_i(x)$where $a_i(x) \neq 0$ when $x \neq 0$. Again let $p(x) = \sum_{i=1}^{t} a_i(x)^2$. $p(x)$ is flat, $p(x) > 0$ when $x \neq 0$ and if $t > 4(n-s)$ we get that $|H(x)| p(x)^{n-s} > \sqrt{\rho(x)}$. For each

$r > s$ let $J(r)$ be the subset of $I(r)$ indexing those essential $r \times r$ minors which corresponds to (essential) submatrices having the upperleft $s \times s$ submatrix as a submatrix. Put $k(x, y) = f(x, y) + \sum_{i=s+1}^{n} y_i^2 p(x)$. This time we find that

$$D_k(x) = H(x)p(x)^{n-s} + \sum_{r=s+1}^{n} \sum_{\alpha \in J(r)} D_\alpha(x)p(x)^{n-r}.$$

Again it is clear that

$$|\sum_{r=s+1}^{n} \sum_{\alpha \in J(r)} D_\alpha(x)p(x)^{n-r}| < \sqrt{\rho(x)},$$

and consequently that $D_k(x) \neq 0$ when $x \neq 0$. So $k(x, y)$ is our wanted representative of the ∞-jet of f.