A SHORT INTRODUCTION TO CONTINUED FRACTIONS

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Abstract. In this talk we introduce continued fractions, prove their basic properties and apply these properties to solve a practical problem. We also state without proof some further properties of continued fractions and provide a brief overview of some facts in this connection. The talk is elementary; it is aimed at undergraduate mathematics majors and mathematics graduate students.

1. Basic Notations

In general, a (simple) continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}},$$

where the letters $a_0, \ a_1, \ a_2, \dots$ denote independent variables, and may be interpreted as one wants (e.g. real or complex numbers, functions, etc.). This expression has precise sense if the number of terms is finite, and may have no meaning for an infinite number of terms. In this talk we only discuss the simplest classical setting.

The letters $a_1, \ a_2, \dots$ denote positive integers. The letter a_0 denotes an integer.

The following standard notation is very convenient.

Notation. *We write*

$$[a_0; a_1, a_2, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n}}}$$

if the number of terms is finite, and

$$[a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

for an infinite number of terms.

Still, in the case of infinite number of terms a certain amount of work must be carried out in order to make the above formula meaningful. At the same time, for the finite number of terms the formula makes sense.

Example 1.

$[-2; 1, 3, 5] = -2+1/(1+1/(3+1/5)) = -2+1/(1+5/16) = -2+1/(21/16) = -2+16/21 = -26/21$

Notation. *For a finite continued fraction $[a_0; a_1, a_2, \dots, a_n]$ and a positive integer $k \le n$, the k-th remainder is defined as the continued fraction*

$$r_k = [a_k; a_{k+1}, a_{k+2}, \dots, a_n].$$

Similarly, for an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ *and a positive integer* k, *the* k*-th remainder is defined as the continued fraction*

$$r_k = [a_k; a_{k+1}, a_{k+2}, \ldots].$$

Thus, at least in the case of a finite continued fraction,

$$\alpha = [a_0; a_1, a_2, \ldots, a_n] = a_0 + 1/(a_1 + 1/(a_2 + \ldots + 1/a_n))$$

we have

$$\alpha = a_0 + 1/(a_1 + 1/(a_2 + \ldots + 1/(a_{k-1} + 1/r_k))) = \text{"}[a_0; a_1, a_2, \ldots, a_{k-1}, r_k]\text{"} \tag{1}$$

for any positive $k \leq n$. Quotation signs appear because we consider the expressions of this kind only with integer entries but the quantity r_k may be a non-integer.

It is not difficult to expand any rational number α into a continued fraction. Indeed, let $a_0 = [\alpha]$ be the greatest integer not exceeding α. Thus the difference $\delta = \alpha - a_0 < 1$ and, of course, $\delta \geq 0$. If $\delta = 0$ then we are done. Otherwise put $r_1 = 1/\delta$, find $a_1 = [r_1]$ and non-negative $\delta = \alpha_1 - a_1 < 1$. Continue the procedure until you obtain $\delta = 0$.

Example 2. *Consider the continued fraction expansion for* $42/31$. *We obtain* $a_0 = [42/31] = 1$, $\delta = 42/31 - 1 = 11/31$. *Now* $r_1 = 1/\delta = 31/11$ *and* $a_1 = [\alpha_1] = [31/11] = 2$. *The new* $\delta = 31/11 - 2 = 9/11$. *Now* $r_2 = 1/\delta = 11/9$ *and* $a_2 = [\alpha_2] = [11/9] = 1$. *It follows that* $\delta = 11/9 - 1 = 2/9$. *Now* $r_3 = 1/\delta = 9/2$ *and* $a_3 = [\alpha_3] = [9/2] = 4$. *It follows that* $\delta = 9/2 - 4 = 1/2$. *Now* $r_4 = 1/\delta = 2$ *and* $a_4 = [\alpha_4] = [2] = 2$. *It follows that* $\delta = 2 - 2 = 0$ *and we are done.*

Thus we have calculated

$$42/31 = [a_0; a_1, a_2, a_3, a_4] = [1; 2, 1, 4, 2].$$

The above example shows that the algorithm stops after finitely many steps. This is in fact quite a general phenomenon. In order to practice with the introduced notations let us prove a simple but important proposition.

Proposition 1. *Any rational number can be represented as a finite continued fraction.*

Proof. By construction, all remainders are positive rationals. For a positive integer k put $r_k = A/B$ and let $a_k = [r_k]$. Then

$$r_k - a_k = \frac{A - Ba_k}{B} := \frac{C}{B}. \tag{2}$$

with $C < B$ because $r_k - a_k < 1$ by construction. If $C = 0$, then the algorithm stops at this point and we are done. Assume now that $C \neq 0$. It follows from (1) that

$$r_k = a_k + \frac{1}{r_{k+1}}. \tag{3}$$

Compare now (2) with (3) to find that

$$r_{k+1} = \frac{B}{C}.$$

Since $C < B$, the rational number r_{k+1} has a denominator which is smaller than the the denominator of the previous remainder r_k. It follows that after a finite number of steps we obtain an integer (a rational with 1 in the denominator) $r_n = a_n$ and the procedure stops at this point.

There appear several natural questions in the connection with Proposition 1.

Is such a continued fraction representation unique? The immediate answer is "no". Here are two "different" continued fraction representations for $1/2$:

$$\frac{1}{2} = [0; 2] = [0; 1, 1].$$

However, we require that $a_n > 1$, where a_n is the last element of a finite continued fraction. Then the answer is "yes".

Exercise 1. *Prove that under the assumption $a_n > 1$ the continued fraction representation given in Proposition 1 is unique. In other words, the correspondence between*

- *finite continued fractions $[a_0; a_1, a_2, \dots a_n]$ with an integer a_0, positive integers a_k for $k > 0$ and $a_n > 1$*

and

- *rational numbers*

is one-to-one.

Hint. Make use of the formulas (5) below.

From now on we assume that $a_n > 1$.

Another natural question is about infinite continued fractions and (as one can easily guess) real numbers. The proof of the corresponding result is slightly more involved, and we do not give it here. In this brief introduction we just formulate the result and refer to the literature ([2, Theorem 14]) for a complete proof. We, however, provide some remarks concerning this result below. In particular, we will explain at some point, what the convergence means.

Theorem 1. *An infinite continued fraction converges and defines a real number. There is a one-to-one correspondence between*

- *all (finite and infinite) continued fractions $[a_0; a_1, a_2, \dots]$ with an integer a_0 and positive integers a_k for $k > 0$ (and the last term $a_n > 1$ in the case of finite continued fractions)*

and

- *real numbers.*

Note that the algorithm we developed above can be applied to any real number and provides the corresponding continued fraction.

Theorem 1 has certain theoretical significance. L.Kronecker (1823-1891) said, "God created the integers; the rest is work of man". Several ways to represent real numbers out of integers are well-known. Theorem 1 provides yet another way to fulfill this task. This way is constructive and at the same time is not tied to any particular base (say to decimal or binary decomposition).

We will discuss some examples later.

2. Main Technical Tool

Truncate finite (or infinite) continued fraction $\alpha = [a_0; a_1, a_2, \dots, a_n]$ at the k-th place (with $k < n$ in the finite case). The rational number $s_k = [a_0; a_1, a_2, \dots, a_k]$ is called the k-th *convergent* of α. Define the integers p_k and q_k by

$$s_k = \frac{p_k}{q_k} \tag{4}$$

written in the reduced form with $q_k > 0$.

The following recursive transformation law takes place.

Theorem 2. *For $k \geq 2$*

$$(5) \qquad \begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \\ q_k &= a_k q_{k-1} + q_{k-2}. \end{aligned}$$

Remark. It does not matter here whether we deal with finite or infinite continued fractions: the convergents are finite anyway.

Proof. We use the induction argument on k. For $k = 2$ the statement is true.

Exercise 2. *Check the assertion of Theorem 2 for $k = 2$.*

Now, assume (5) for $2 \leq k < l$. Let

$$\alpha = [a_0; a_1, a_2, \dots a_l] = \frac{p_l}{q_l}$$

be an arbitrary continued fraction of length $l + 1$. We denote by p_r/q_r the r-th convergent α. Consider also the continued fraction

$$\beta = [a_1; a_2, \dots, a_l]$$

and denote by p'_r/q'_r its r-th convergent. We have $\alpha = a_0 + 1/\beta$ which translates as

$$(6) \qquad \begin{aligned} p_l &= a_0 p'_{l-1} + q'_{l-1} \\ q_l &= p'_{l-1}. \end{aligned}$$

Also, by the induction assumption,

$$(7) \qquad \begin{aligned} p'_{l-1} &= a_l p'_{l-2} + p'_{l-3} \\ q'_{l-1} &= a_l q'_{l-2} + q'_{l-3} \end{aligned}$$

Combining (6) and (7) we obtain the formulas

$$p_l = a_0(a_l p'_{l-2} + p'_{l-3}) + a_l q'_{l-2} + q'_{l-3} = a_l(a_0 p'_{l-2} + q'_{l-2}) + (a_0 p'_{l-3} + q'_{l-3}) = a_l p_{l-1} + p_{l-2}$$

and

$$q_l = a_l p'_{l-2} + p'_{l-3} = a_l q_{l-1} + q_{l-2},$$

which complete the induction step. We have thus proved that

$$s_k = \frac{p_k}{q_k},$$

where p_k and q_k are defined by the recursive formulas (5). We still have to check that these are the quantities defined by (4), namely that $q_k > 0$ and that q_k and p_k are coprime. The former assertion follows from (5) since $a_k > 0$ for $k > 0$. To prove the latter assertion, multiply the equations (5) by q_{k-1} and p_{k-1} respectively and subtract them. We obtain

$$(8) \qquad p_k q_{k-1} - q_k p_{k-1} = -(p_{k-1} q_{k-2} - q_{k-1} p_{k-2}).$$

Exercise 3. *Check that for $k = 2$*

$$p_2 q_1 - q_2 p_1 = -1.$$

Hint. Introduce formally $p_{-1} = 1$ and $q_{-1} = 0$, check that then formulas 5 are true also for $k = 1$.

Exercise 4. *Combine the previous exercises with (8) to obtain*

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k$$

for $k \geq 1$. *Derive from this that* q_k *and* p_k *are coprime.*

This concludes the proof of Theorem 5.

3. Some Inequalities.

As an immediate consequence of (5) we find that

$$\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k-1}} \tag{9}$$

and

$$\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = \frac{(-1)^{k-1} a_k}{q_k q_{k-2}}.$$

Since all the numbers q_k and a_k are positive, the above formulas imply the following.

Proposition 2. *The subsequence of convergents* p_k/q_k *for even indices* k *is increasing.*
The subsequence of convergents p_k/q_k *for odd indices* k *is decreasing.*
Every convergent with an odd index is bigger than every convergent with an even index.

Exercise 5. *Prove Proposition 2*

Remark. Proposition 2 implies that both subsequences of convergents (those with odd indices and those with even indices) have limits. This is a step towards making sense out of an infinite continued fraction: this should be *common* limit of these two subsequences. It is somehow more technically involved (although still fairly elementary!) to prove that these two limits coincide.

Theorem 3. *Let* $\alpha = [a_0; a_1, a_2, \ldots, a_n]$. *For* $k < n$ *we have*

$$\frac{1}{q_k(q_{k+1} + q_k)} \leq \left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k q_{k+1}}$$

Proof.

Exercise 6. *Combine (9) with Proposition 2 to prove the inequality*

$$\left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k q_{k+1}}.$$

Another inequality, which provides the lower bound for the distance between the number α and k-th convergent is slightly more involved. To prove it we first consider the following way to add fractions which students sometimes prefer.

Definition. *The number*

$$\frac{a + c}{b + d}$$

is called the mediant of the two fractions a/b *and* c/d. *(The quantities* a, b, c *and* d *are integers.)*

Lemma 1. *If*

$$\frac{a}{b} \leq \frac{c}{d}$$

then

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}.$$

Exercise 7. *Prove Lemma 1*

Consider now the sequence of fractions

$$\frac{p_k}{q_k},\ \frac{p_k+p_{k+1}}{q_k+q_{k+1}},\ \frac{p_k+2p_{k+1}}{q_k+2q_{k+1}},\dots,\frac{p_k+a_kp_{k+1}}{q_k+a_kq_{k+1}}=\frac{p_{k+2}}{q_{k+2}}, \tag{10}$$

where the last equality follows from (5).

Exercise 8. *Use (5) to show that the sign of the difference between two consecutive fractions in (10) depends only on the parity of k.*

It follows that the sequence (10) is increasing if k is even and is decreasing if k is odd. Thus, in particular, the fraction

$$\frac{p_k+p_{k+1}}{q_k+q_{k+1}} \tag{11}$$

is between the quantities p_k/q_k and α. Therefore the distance between p_k/q_k and the fraction (11) is smaller than the distance between p_k/q_k and α:

$$\left|\alpha-\frac{p_k}{q_k}\right| \geq \frac{p_k+p_{k+1}}{q_k+q_{k+1}} = \frac{1}{q_k(q_k+q_{k+1})}.$$

The second (right) inequality in Theorem 3 is now proved. This finishes the proof of Theorem 3.

4. Very Good Approximation.

Continued fractions provide a representation of numbers which is, in a sense, generic and canonical. It does not depend on an arbitrary choice of a base. Such a representation should be the best in a sense. In this section we quantify this naive idea.

Definition. *A rational number a/b is refered to as a "good" approximation to a number α if*

$$\frac{c}{d} \neq \frac{a}{b} \quad \textit{and} \quad 0<d\leq b$$

imply

$$|d\alpha - c| > |b\alpha - a|.$$

Remarks. 1. Our "good approximation" is "the best approximation of the second kind" in a more usual terminology.
2. Although we use this definition only for rational α, it may be used for any real α as well. Neither the results of this section nor the proofs alter.
3. Naively, this definition means that a/b approximates α better then any other rational number whose denominator does not exceed b. There is another, more common, definition of "the best approximation". A rational number x/y is refered to as "the best approximation of the first kind" if $c/d \neq x/y$ and $0 < d \leq y$ imply $|\alpha - c/d| > |\alpha - x/y|$. In other words, x/y is closer to α than any rational number whose denominator does not exceed y. In our definition we consider

a slightly different measure of approximation, which takes into the account the denominator, namely $b|\alpha - a/b| = |b\alpha - a|$ instead of taking just the distance $|\alpha - a/b|$.

Theorem 4. *Any "good" approximation is a convergent.*

Proof. Let a/b be a "good" approximation to $\alpha = [a_0; a_1, a_2, \ldots, a_n]$. We have to prove that $a/b = p_k/q_k$ for some k.

Exercise 9. *Prove that if a/b is a "good" approximation then $a/b \geq a_0$.*

Thus we have $a/b > p_1/q_1$ or a/b lies between two consecutive convergents p_{k-1}/q_{k-1} and p_{k+1}/q_{k+1} for some k. Assume the latter. Then

$$\left|\frac{a}{b} - \frac{p_{k-1}}{q_{k-1}}\right| \geq \frac{1}{bq_{k-1}}$$

and

$$\left|\frac{a}{b} - \frac{p_{k-1}}{q_{k-1}}\right| < \left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| = \frac{1}{q_k q_{k-1}}.$$

It follows that

$$b > q_k. \tag{12}$$

Also

$$\left|\alpha - \frac{a}{b}\right| \geq \left|\frac{p_{k+1}}{q_{k+1}} - \frac{a}{b}\right| \geq \frac{1}{bq_{k+1}},$$

which implies

$$|b\alpha - a| \geq \frac{1}{q_{k+1}}.$$

At the same time Theorem 3 (it right inequality multiplied by q_k) reads

$$|q_k\alpha - p_k| \leq \frac{1}{q_{k+1}}.$$

It follows that

$$|q_k\alpha - p_k| \leq |b\alpha - a|,$$

and the latter inequality together with (12) show that a/b is not a "good" approximation of α in this case.

Exercise 10. *Show that if $a/b > p_1/q_1$ then a/b is not a "good" approximation to α.*

This finishes the proof of Theorem 4.

5. An Application.

Consider the following problem which may be of certain practical interest. Assume that we calculate certain quantity using a computer. Also assume that we know in advance that the quantity in question is a rational number. The computer returns a decimal which has high accuracy and is pretty close to our desired answer. How to guess the exact answer?

To be more specific consider an example.

Example 3. *Assume that the desired answer is*

$$\frac{123456}{121169}$$

and the result of computer calculation with a modest error of 10^{-15} *is*

$\alpha = 123456/121169 + 10^{-15} =$
$1.01887446459077916933374047817511079566555802226642127937013592$
$58556231379313190667579991582005298384900428327377464533007617 45$
$99113634675535821868629765038912593155014896549447465936006734 39576129207$

with some two hundred digits of accuracy which, of course come short to help in guessing the period and the exact denominator of 121169.

Solution. Since $123456/121169$ is a good (just in a naive sense) approximation to α, it should be among its convergents. This is not an exact statement, but it offers a hope! We have

$$\alpha = [1; 52, 1, 53, 2, 4, 1, 2, 1, 68110, 4, 1, 2, 106, 22, 3, 1, 1, 10, 2, 1, 3, 1, 3, 4, 2, 11].$$

We are not going to check all convergents, because we notice the irregularity: one element, 68110 is far mor than the others. In order to explain this we use the left inequality from Theorem 3 together with the formula (5). Indeed, we have an approximation of α which is unexpectedly good: $|\alpha - p_k/q_k|$ is very small (it is around 10^{-15}) and with a modest q_k too. We have

$$q_k(q_{k+1} + q_k) = q_k(a_{k+1}q_k + q_{k-1}) = q_k^2(a_{k+1} + q_{k-1}/q_k)$$

and

$$\left|\alpha - \frac{p_k}{q_k}\right| \geq \frac{1}{q_k^2(a_{k+1} + q_{k-1}/q_k)}.$$

It follows that $1/q_k^2(a_{k+1} + q_{k-1}/q_k)$ is small (smaller than 10^{-15}) and therefore, a_{k+1} should be big. This is exactly what we see. Of course, our guess is correct:

$$\frac{123456}{121169} = [1, 52, 1, 53, 2, 4, 1, 2, 1].$$

In this way we conclude that in general an unexpectedly big element allows to cut the continued fraction (right before this element) and to guess the exact rational quantity. There is probably no need (although this is, of course, possible) to quantify this procedure. I prefer to use it just for guessing the correct quantities on the spot from the first glance.

6. Further Overview.

Being a very natural object, continued fractions appear in many areas of Mathematics, sometimes in an unexpected way. The Dutch mathematician and astronomer, Christian Huygens (1629-1695), made the first practical application of the theory of "anthyphaeiretic ratios" (the old name of continued fractions) in 1687. He wrote a paper explaining how to use convergents to find the best rational approximations for gear ratios. These approximations enabled him to pick the gears with the best numbers of teeth. His work was motivated by his desire to build a mechanical planetarium. Further continued fractions attracted attention of most prominent mathematicians. Euler, Jacobi, Cauchy, Gauss and many others worked with the subject. Continued fractions find their applications in some areas of contemporary Mathematics. There are mathematicians who continue to develop the theory of continued fractions nowadays, The Australian mathematician A.J. van der Poorten is, probably, the most prominent among them.

In conclusion of this brief overview let me formulate some facts which will make the picture of the elementary theory slightly more complete.

First of all, I would like to return to the infinite continued fractions now. We know that decimal fractions which represent rational numbers may be finite or periodic. Theorem 1 establishes a one-to-one correspondence between rational numbers and finite continued fractions. The following result answers in a very elegant way the question about the periodic continued fractions (see [2, Theorem 28] for the proof which is still elementary).

Theorem 5. (Lagrange) *There is a one-to one correspondence between quadratic irrationalities and periodic continued fractions.*

Example 4. *The Golden Ratio*

$$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, 1, 1, \ldots],$$

which suggests an intimate connection between the continued fractions and Fibonacci numbers.

Exercise 11. *Formulate and prove the connection, mentioned above, with Fibonacci numbers.*

Hint. The ratios of consecutive Fibonacci numbers are rational approximations of the Golden Ratio.

Example 5.

$$\sqrt{29} = [5; 2, 1, 1, 2, 10, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10, \ldots].$$

Amazingly, in contrast to the perfect result of Theorem 5, not much is known about the continued fraction decomposition of other algebraic numbers.

Even more surprisingly, we know something about the continued fractions decompositions of some transcendental numbers (see [3] for an elegant proof of the identity below and further development in this direction)

Example 6.

$$e - 1 = [1; 1, 2, 1, 1, 4, 1, 1, 6, \ldots] = \overline{[1, 1, 2h]}_{h=1}^{\infty}, \tag{13}$$

where $e = 2.71828\ldots$.

7. A Formula of Gauss, a Theorem of Kuzmin and Lévi and a Problem of Arnold.

Although Theorem 5 and formula (13) provide certain regular expressions, it is intuitively clear that for an arbitrary chosen irrational number the elements of its continued fraction appear without any regularity. In this connection Gauss asked about a probability c_k for a number k to appear as an element of a continued fraction. Such a probability is defined in a natural way: as a limit when $N \to \infty$ of the number of occurences of k among the first N elements of the continued fraction expension. Moreover, Gauss provided an answer, but never published the proof. Two different proofs were found independetly by R.O.Kuzmin (1928) and P. Lévy (1929) (see [2] for a detailed exposition of the R.O.Kuzmin's proof).

Theorem 6. *For almost every real α the probability for a number k to appear as an element in the continued fraction expansion of α is*

$$c_k = \frac{1}{\ln 2} \ln\left(1 + \frac{1}{k(k+2)}\right). \tag{14}$$

Remarks. 1. The words "for almost every α" mean that the measure of the set of exceptions is zero.
2. Even the existence of p_k (defined as a limit) is highly non-trivial.

Exercise 12. *Prove that c_k really define a probability distribution, namely that*

$$\sum_{k=1}^{\infty} c_k = 1.$$

Theorem 6 may (and probably should) be considered as a result from ergodic theory rather than number theory. This constructs a bridge between these two areas of Mathematics and explains the recent attention to continued fractions of the mathematicians who study dynamical systems. In particular, V.I.Arnold formulated the following open problem. Consider the set of pairs of integers (a, b) such that the corresponding points on the plane are contained in a quarter of a circle of radii N:

$$a^2 + b^2 \leq N^2.$$

Expand the numbers p/q into continued fractions and compute the frequences s_k for the appearence of k in these fractions. Do these frequences have limits as $N \to \infty$? If so, do these limits have anything to do with the probabilities, given by (14)? These questions demand nothing but experimental computer investigation, and such an experiment may be undertaken by a student. Of course, it would be extremely challenging to find a phenomena experimentally in this way and to prove it after that theoretically.

8. Concluding Remark.

Of course, one can consider more general kinds of continued fractions. In particular, one may ease the assumption that the elements are positive integers and consider, allowing arbitrary reals as the elements (the question of convergence may usually be solved). The following identities were discovered independently by three prominent mathematicians. The English mathematician R.J. Rogers found and proved these identities in 1894, Ramanujan found the identities (without proof) and formulated them in his letter to Hardy from India in 1913. Independently, being separated from England by the war, I. J. Schur found the identities and published two different proofs in 1917. We refer an interested reader to [1] for a detailed discussion and just state the amazing identities here.

$$[0; e^{-2\pi}, e^{-4\pi}, e^{-6\pi}, e^{-8\pi}, \ldots] = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}$$

$$[1; e^{-\pi}, e^{-2\pi}, e^{-3\pi}, e^{-4\pi}, \ldots] = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right) e^{\pi/5}$$

References

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