Congruence Varieties

Ralph Freese

Hawaii

Conference on Lattice Theory
In honour of the 70th birthday of
George Grätzer
and
E. Tamás Schmidt
Congratulations!!

Congratulations to

George Grätzer
and
E. Tamás Schmidt
Congratulations!!

Congratulations to

George Grätzer
and
E. Tamás Schmidt
on their
Congratulations!!

Congratulations to

George Grätzer
and
E. Tamás Schmidt
on their

Beating the alternative!!
\( \mathcal{K} \) is a variety of algebras.
Notation

- $\mathcal{K}$ is a variety of algebras.
- $\text{Con}(\mathcal{K}) = \{\text{Con}(A) : A \in \mathcal{K}\}$.
\( \mathcal{K} \) is a variety of algebras.

\( \text{Con}(\mathcal{K}) = \{ \text{Con} (A) : A \in \mathcal{K} \} \).

\( \text{VCon} (\mathcal{K}) \) is the \textit{congruence variety} associated with \( \mathcal{K} \).
Notation

- \( \mathcal{K} \) is a variety of algebras.
- \( \text{Con}(\mathcal{K}) = \{ \text{Con}(A) : A \in \mathcal{K} \} \).
- \( \text{VCon}(\mathcal{K}) \) is the congruence variety associated with \( \mathcal{K} \).

Lemma

\( \text{PCon}(\mathcal{K}) \subseteq \text{SCon}(\mathcal{K}) \).
So \( \text{VCon}(\mathcal{K}) \subseteq \text{HSCon}(\mathcal{K}) \).
Beginnings

Theorem (Nation)
There are varieties of lattices which are not congruence varieties.
There is a lattice equation $\sigma$ which does not imply modularity but if $\text{Con}(K)$ satisfies $\sigma$, then it is modular.

Problem
Which varieties of lattices are congruence varieties?
Are there any besides the four obvious ones: distributive lattices, the congruence variety of groups, and the two trivial varieties?

The work in congruence varieties over the next several years was stimulated by McKenzie's conjecture:

Conjecture (McKenzie, 1973)
There are no nonmodular congruence varieties other than the variety of all lattices.
Theorem (Nation)

There are varieties of lattices which are not congruence varieties.
Theorem (Nation)

There are varieties of lattices which are not congruence varieties. There is a lattice equation $\sigma$ which does not imply modularity but if $\text{Con} (\mathcal{K})$ satisfies $\sigma$, then it is modular.
Beginnings

Theorem (Nation)

There are varieties of lattices which are not congruence varieties. There is a lattice equation $\sigma$ which does not imply modularity but if $\text{Con}(\mathcal{K})$ satisfies $\sigma$, then it is modular.

Problem

Which varieties of lattices are congruence varieties?
Theorem (Nation)

There are varieties of lattices which are not congruence varieties. There is a lattice equation $\sigma$ which does not imply modularity but if $\text{Con}(\mathcal{K})$ satisfies $\sigma$, then it is modular.

Problem

Which varieties of lattices are congruence varieties? Are there any besides the four obvious ones: distributive lattices, the congruence variety of groups, and the two trivial varieties?
Theorem (Nation)

There are varieties of lattices which are not congruence varieties. There is a lattice equation $\sigma$ which does not imply modularity but if $\text{Con}(\mathcal{K})$ satisfies $\sigma$, then it is modular.

Problem

Which varieties of lattices are congruence varieties? Are there any besides the four obvious ones: distributive lattices, the congruence variety of groups, and the two trivial varieties?

The work in congruence varieties over the next several years was stimulated by McKenzie’s conjecture:
Theorem (Nation)

There are varieties of lattices which are not congruence varieties. There is a lattice equation $\sigma$ which does not imply modularity but if $\text{Con}(\mathcal{K})$ satisfies $\sigma$, then it is modular.

Problem

Which varieties of lattices are congruence varieties? Are there any besides the four obvious ones: distributive lattices, the congruence variety of groups, and the two trivial varieties?

The work in congruence varieties over the next several years was stimulated by McKenzie’s conjecture:

Conjecture (McKenzie, 1973)

There are no nonmodular congruence varieties other than the variety of all lattices.
Suppose $N_5 \leq \text{Con} (A)$:
Suppose $N_5 \leq \text{Con}(A)$: Then in $\text{Con}(A(\alpha))$
Suppose \( N_5 \leq \text{Con} (A) \): Then in \( \text{Con}(A(\alpha)) \)
Suppose $N_5 \leq \text{Con} (A)$: Then in $\text{Con}(A(\alpha))$
An Amalgamation Technique

Suppose $N_5 \leq \text{Con}(A)$: Then in $\text{Con}(A(\alpha))$

So this proves Nation’s Theorem (at least as stated here).
An Amalgamation Technique

Suppose $N_5 \leq \text{Con} (A)$: Then in $\text{Con}(A(\alpha))$

- So this proves Nation’s Theorem (at least as stated here).
- This same amalgamation-like technique is the basis of
Suppose $N_5 \leq \text{Con}(A)$: Then in $\text{Con}(A(\alpha))$

- So this proves Nation’s Theorem (at least as stated here).
- This same amalgamation-like technique is the basis of

**Theorem (with Jónsson)**

*Every modular congruence variety is arguesian.*
Minimal Modular Congruence Varieties

- It can also prove

\[ \text{Theorem (with C. Herrmann and A. Huhn)} \]

Every modular, nondistributive congruence variety contains \( M_p \) for some \( p \) a prime or \( 0 \), where \( M_p \) is the congruence variety of vector spaces over the prime field of characteristic \( p \).

Idea of Proof:

There is an abelian congruence \( \alpha \succ 0 \) in some algebra \( A \) in the variety.

Let \( B = \{ (a_1, \ldots, a_n) \in A^n : a_i \alpha a_j \} \).

The lattice of subspaces of a vector space embeds into \( \text{Con}(B) \).
It can also prove

**Theorem (with C. Herrmann and A. Huhn)**

*Every modular, nondistributive congruence variety contains $\mathcal{M}_p$ for some $p$ a prime or 0,*
It can also prove

Theorem (with C. Herrmann and A. Huhn)

Every modular, nondistributive congruence variety contains $\mathcal{M}_p$ for some $p$ a prime or 0,

where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$. 
It can also prove

**Theorem (with C. Herrmann and A. Huhn)**

- Every modular, nondistributive congruence variety contains $\mathcal{M}_p$ for some $p$ a prime or 0,
- where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.

**Idea of Proof:**
It can also prove

**Theorem (with C. Herrmann and A. Huhn)**

*Every modular, nondistributive congruence variety contains $\mathcal{M}_p$ for some $p$ a prime or 0,*

*where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.*

**Idea of Proof:**

*There is an abelian congruence $\alpha \succ 0$ in some algebra $A$ in the variety.*
It can also prove

**Theorem (with C. Herrmann and A. Huhn)**

Every modular, nondistributive congruence variety contains $\mathcal{M}_p$ for some $p$ a prime or 0,

where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.

**Idea of Proof:**

- There is an abelian congruence $\alpha \succ 0$ in some algebra $A$ in the variety.
- Let $B = \{(a_1, \ldots, a_n) \in A^n : a_i \alpha a_j\}$.
It can also prove

**Theorem (with C. Herrmann and A. Huhn)**

Every modular, nondistributive congruence variety contains $\mathcal{M}_p$ for some $p$ a prime or 0,

where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.

**Idea of Proof:**

- There is an abelian congruence $\alpha > 0$ in some algebra $A$ in the variety.
- Let $B = \{(a_1, \ldots, a_n) \in A^n : a_i \alpha a_j\}$.
- The lattice of subspaces of a vector space embeds into $\text{Con} (B)$.
There is a variety $\mathcal{P}$ such that $\text{VCon}(\mathcal{P})$ is a nonmodular, proper congruence variety.
There is a variety $P$ such that $\text{VCon}(P)$ is a nonmodular, proper congruence variety.

Every nonmodular congruence variety contains $\text{Con}(P)$. 
Polin’s Variety $\mathcal{P}$

**Theorem**

There is a variety $\mathcal{P}$ such that $\text{VCon}(\mathcal{P})$ is a nonmodular, proper congruence variety.

**Theorem (with A. Day)**

Every nonmodular congruence variety contains $\text{Con}(\mathcal{P})$.

**Theorem (with G. Czédli)**

- There is an algorithm to decide if a lattice equation implies congruence modularity.
- And one to decide congruence distributivity.
Let $\mathcal{K}_R$ be the variety of unitary left $R$-modules.
Let $\mathcal{K}_R$ be the variety of unitary left $R$-modules.
Let $f(R, p)$ be the least $k$ such that $R\rho^{k+1} = R\rho^k$; or $\omega$ is there is no such $k$. 

Theorem (Czédli, Hutchinson)
The congruence varieties for $\mathcal{K}_R$ and $\mathcal{K}_S$ are the same if and only if $R$ and $S$ have the same characteristic $n$, and, if $n = 0$, $f(R, p) = f(S, p)$ for all $p$. 

Theorem (Day, Kiss)
If $\mathcal{K}$ is residually small, congruence modular but not distributive, then its congruence variety is one of the above. 

Theorem (Pálfy, Szabó)
The congruence varieties of groups and of abelian groups are distinct.
Let $\mathcal{K}_R$ be the variety of unitary left $R$-modules. Let $f(R, p)$ be the least $k$ such that $R\rho^{k+1} = R\rho^k$; or $\omega$ is there is no such $k$.

**Theorem (Czédli, Hutchinson)**

*The congruence varieties for $\mathcal{K}_R$ and $\mathcal{K}_S$ are the same if and only if*

Ralph Freese (Hawaii)
Let \( \mathcal{K}_R \) be the variety of unitary left \( R \)-modules.
Let \( f(R, p) \) be the least \( k \) such that \( R^p^{k+1} = R^p^k \); or \( \omega \) is there is no such \( k \).

**Theorem (Czédli, Hutchinson)**

The congruence varieties for \( \mathcal{K}_R \) and \( \mathcal{K}_S \) are the same if and only if
- \( R \) and \( S \) have the same characteristic \( n \),
Let $\mathcal{K}_R$ be the variety of unitary left $R$-modules. Let $f(R, p)$ be the least $k$ such that $Rp^{k+1} = Rp^k$; or $\omega$ is there is no such $k$.

**Theorem (Czédli, Hutchinson)**

The congruence varieties for $\mathcal{K}_R$ and $\mathcal{K}_S$ are the same if and only if

- $R$ and $S$ have the same characteristic $n$,
- and, if $n = 0$, $f(R, p) = f(S, p)$ for all $p$. 

**Theorem (Day, Kiss)**

If $\mathcal{K}$ is residually small, congruence modular but not distributive, then its congruence variety is one of the above.

**Theorem (Pálfy, Szabó)**

The congruence varieties of groups and of abelian groups are distinct.
Let $\mathcal{K}_R$ be the variety of unitary left $R$-modules.
Let $f(R, p)$ be the least $k$ such that $R^{p^{k+1}} = R^{p^k}$; or $\omega$ is there is no such $k$.

Theorem (Czédli, Hutchinson)

The congruence varieties for $\mathcal{K}_R$ and $\mathcal{K}_S$ are the same if and only if

- $R$ and $S$ have the same characteristic $n$,
- and, if $n = 0$, $f(R, p) = f(S, p)$ for all $p$.

Theorem (Day, Kiss)

If $\mathcal{K}$ is residually small, congruence modular but not distributive, then its congruence variety is one of the above.
Let $K_R$ be the variety of unitary left $R$-modules. Let $f(R, p)$ be the least $k$ such that $Rp^{k+1} = Rp^k$; or $\omega$ is there is no such $k$.

**Theorem (Czédli, Hutchinson)**

The congruence varieties for $K_R$ and $K_S$ are the same if and only if

- $R$ and $S$ have the same characteristic $n$,
- and, if $n = 0$, $f(R, p) = f(S, p)$ for all $p$.

**Theorem (Day, Kiss)**

If $K$ is residually small, congruence modular but not distributive, then its congruence variety is one of the above.

**Theorem (Pálfy, Szabó)**

The congruence varieties of groups and of abelian groups are distinct.
This is the title of the long awaited monograph of Keith Kearnes and Emil Kiss. It is a fundamental study of the relationships between

Maltsev conditions, Term conditions, The shape of congruence lattices.

A few highlights:
- Does not assume the varieties are locally finite.
- Strong theory of solvability for varieties satisfying a congruence identity:
  - Congruences in solvable intervals permute.
  - Transposes of abelian (solvable) intervals are abelian (solvable).
- Con \((A)\) has a SD by modular decomposition.
This is the title of the long awaited monograph of Keith Kearnes and Emil Kiss. It is a fundamental study of the relationships between

- Maltsev conditions,
- Term conditions,
- The shape of congruence lattices.
This is the title of the long awaited monograph of **Keith Kearnes** and **Emil Kiss**. It is a fundamental study of the relationships between

- Maltsev conditions,
- Term conditions,
- The shape of congruence lattices.

A few highlights:

- Does not assume the varieties are locally finite.
“The Shape of Congruence Lattices”

This is the title of the long awaited monograph of Keith Kearnes and Emil Kiss. It is a fundamental study of the relationships between

- Maltsev conditions,
- Term conditions,
- The shape of congruence lattices.

A few highlights:

- Does not assume the varieties are locally finite.
- Strong theory of solvability for varieties satisfying a congruence identity:
  - Congruences in solvable intervals permute.
  - Transposes of abelian (solvable) intervals are abelian (solvable).
  - $\text{Con}(A)$ has a SD by modular decomposition.
Theorem

If a congruence variety contains every $N_k$ then it is the variety of all lattices.
Theorem

If a congruence variety contains every $N_k$ then it is the variety of all lattices.
Theorem

If a congruence variety contains every $N_k$ then it is the variety of all lattices.
Theorem

If a congruence variety contains every $N_k$ then it is the variety of all lattices.
The full version of the theorem above.

Theorem

Let $\mathcal{K}$ be a variety. TFAE
The full version of the theorem above.

**Theorem**

Let $\mathcal{K}$ be a variety. TFAE

- $\mathcal{K}$ satisfies a nontrivial congruence identity.
The full version of the theorem above.

**Theorem**

Let $\mathcal{K}$ be a variety. TFAE

- $\mathcal{K}$ satisfies a nontrivial congruence identity.
- $\mathcal{K}$ satisfies an idempotent Maltsev condition which fails in the variety of semilattices.

**Corollary**

There is no largest, proper congruence variety.
Theorem

Let $\mathcal{K}$ be a variety. TFAE

- $\mathcal{K}$ satisfies a nontrivial congruence identity.
- $\mathcal{K}$ satisfies an idempotent Maltsev condition which fails in the variety of semilattices.
- $\mathcal{K}$ has a Hobby-McKenzie term.

Corollary

There is no largest, proper congruence variety.
The full version of the theorem above.

**Theorem**

Let $\mathcal{K}$ be a variety. TFAE

- $\mathcal{K}$ satisfies a nontrivial congruence identity.
- $\mathcal{K}$ satisfies an idempotent Maltsev condition which fails in the variety of semilattices.
- $\mathcal{K}$ has a Hobby-McKenzie term.
- There is a $k$ such that $N_k$ cannot be embedded in $\text{Con}(A)$ for any $A \in \mathcal{K}$.

**Corollary**

There is no largest, proper congruence variety.
The full version of the theorem above.

**Theorem**

Let \( \mathcal{K} \) be a variety. TFAE

- \( \mathcal{K} \) satisfies a nontrivial congruence identity.
- \( \mathcal{K} \) satisfies an idempotent Maltsev condition which fails in the variety of semilattices.
- \( \mathcal{K} \) has a Hobby-McKenzie term.
- There is a \( k \) such that \( N_k \) cannot be embedded in \( \text{Con} (A) \) for any \( A \in \mathcal{K} \).

**Corollary**

There is no largest, proper congruence variety.
Theorem

Suppose $\mathcal{K}$ satisfies a nontrivial congruence identity. Then every SD$_\vee$-failure interval is abelian.
Theorem

Suppose $\mathcal{K}$ satisfies a nontrivial congruence identity. Then every $SD_\vee$-failure interval is abelian.

- This and other results from the monograph and from Kearnes Szendrei give
Theorem

If $\mathcal{K}$ is a variety which is not congruence semidistributive, then $\mathcal{V}_{\text{Con}}(\mathcal{K})$ contains $M_p$ for some $p$ a prime or $0$, where $M_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$. 

Idea of Proof:
There is an abelian congruence $\alpha \succeq 0$ in some algebra $A$ in the variety. Let $B = \{(a_1, \ldots, a_n) \in A^n : a_i \alpha a_j\}$. The lattice of subspaces of a vector space embeds into $\mathcal{V}_{\text{Con}}(B)$. 
If $\mathcal{K}$ is a variety which is not congruence semidistributive, then $\text{VCon} (\mathcal{K})$ contains $\mathcal{M}_p$ for some $p$ a prime or 0.
<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $\mathcal{K}$ is a variety which is not congruence semidistributive, then $\text{VCon} (\mathcal{K})$ contains $\mathcal{M}_p$ for some $p$ a prime or $0$, where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.</td>
</tr>
</tbody>
</table>

Idea of Proof:

There is an abelian congruence $\alpha \succ 0$ in some algebra $A$ in the variety. Let $B = \{(a_1, \ldots, a_n) \in A^n : a_i \alpha a_j\}$. The lattice of subspaces of a vector space embeds into $\text{Con} (B)$. 

Ralph Freese (Hawaii) 
Congruence Varieties 
June 7, 2006 13 / 22
If $\mathcal{K}$ is a variety which is not congruence semidistributive, then $\mathcal{VCon}(\mathcal{K})$ contains $\mathcal{M}_p$ for some $p$ a prime or 0, where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.

Idea of Proof:
Theorem

If $\mathcal{K}$ is a variety which is not congruence semidistributive, then $\text{VCon}(\mathcal{K})$ contains $\mathcal{M}_p$ for some $p$ a prime or $0$, where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.

Idea of Proof:

- There is an abelian congruence $\alpha \succ 0$ in some algebra $A$ in the variety.
If $\mathcal{K}$ is a variety which is not congruence semidistributive, then $\text{VCon}(\mathcal{K})$ contains $\mathcal{M}_p$ for some $p$ a prime or 0, where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.

**Idea of Proof:**

- There is an abelian congruence $\alpha > 0$ in some algebra $A$ in the variety.
- Let $B = \{(a_1, \ldots, a_n) \in A^n : a_i \alpha a_j\}$. 
Theorem

If $\mathcal{K}$ is a variety which is not congruence semidistributive, then $VCon(\mathcal{K})$ contains $\mathcal{M}_p$ for some $p$ a prime or $0$, where $\mathcal{M}_p$ is the congruence variety of vector spaces over the prime field of characteristic $p$.

Idea of Proof:

- There is an abelian congruence $\alpha \succ 0$ in some algebra $A$ in the variety.
- Let $B = \{(a_1, \ldots, a_n) \in A^n : a_i \alpha a_j\}$.
- The lattice of subspaces of a vector space embeds into $Con(B)$. 
Define lattice terms in variables $x$, $y$ and $z$ by

- $y^0 = y$, $z^0 = z$, 

Theorem A variety $K$ satisfies a nontrivial congruence identity iff, for some $m$, it satisfies the congruence identity $z^m = z^{m+1}$.

Theorem Let $K$ be a variety. TFAE $K$ satisfies a nontrivial congruence identity.

- $D_2$ is not a sublattice of a congruence lattice of a member of $K$.
- $D_2$ is not in the congruence variety of $K$. 

Ralph Freese (Hawaii) Congruence Varieties June 7, 2006 14 / 22
Herringbone Terms

Define lattice terms in variables $x$, $y$ and $z$ by

- $y^0 = y$, $z^0 = z$,
- $y^{n+1} = y \land (x \lor z^n)$  \quad  $z^{n+1} = z \land (x \lor y^n)$
Define lattice terms in variables $x$, $y$ and $z$ by

- $y^0 = y$, $z^0 = z$,
- $y^{n+1} = y \land (x \lor z^n)$
- $z^{n+1} = z \land (x \lor y^n)$

**Theorem**

A variety $\mathcal{K}$ satisfies a nontrivial congruence identity iff, for some $m$, it satisfies the congruence identity

$$z^m = z^{m+1}.$$
Herringbone Terms

Define lattice terms in variables $x$, $y$ and $z$ by

- $y^0 = y$, $z^0 = z$,
- $y^{n+1} = y \land (x \lor z^n)$, $z^{n+1} = z \land (x \lor y^n)$

**Theorem**

A variety $\mathcal{K}$ satisfies a nontrivial congruence identity iff, for some $m$, it satisfies the congruence identity

$$z^m = z^{m+1}.$$ 

**Theorem**

Let $\mathcal{K}$ be a variety. TFAE

- $\mathcal{K}$ satisfies a nontrivial congruence identity.
- $D_2$ is not a sublattice of a congruence lattice of a member of $\mathcal{K}$.
- $D_2$ is not in the congruence variety of $\mathcal{K}$.
Let $\mathcal{K}$ is a variety with congruence variety $\mathcal{V}$. Suppose $\mathcal{V} \neq \mathcal{L}$ and $M_3 \in \mathcal{V}$. Then $M_3$ is projective in $\mathcal{V}$. 

Proof. $\mathcal{V}(3)$ is a sublattice of $\text{Con}(A)$, for some $A \in \mathcal{K}$, let $\alpha, \beta, \text{ and } \gamma$ be the free generators. By the above theorem, $\beta m = \beta m + 1$ and $\gamma m = \gamma m + 1$, for some $m$, which implies $\alpha \vee \beta m = \alpha \vee \gamma m$. So the interval from $\alpha \vee (\beta m \wedge \gamma m)$ to $\alpha \vee \beta m$ is abelian.
Theorem

Let \( \mathcal{K} \) is a variety with congruence variety \( \mathcal{V} \). Suppose \( \mathcal{V} \neq \mathcal{L} \) and \( M_3 \in \mathcal{V} \). Then \( M_3 \) is projective in \( \mathcal{V} \).

Proof.

- \( F_\mathcal{V}(3) \) is a sublattice of \( \text{Con} (\mathcal{A}) \), for some \( \mathcal{A} \in \mathcal{K} \),
An Application

Theorem

Let $\mathcal{K}$ is a variety with congruence variety $\mathcal{V}$. Suppose $\mathcal{V} \neq \mathcal{L}$ and $M_3 \in \mathcal{V}$. Then $M_3$ is projective in $\mathcal{V}$.

Proof.

- $F_\mathcal{V}(3)$ is a sublattice of $\text{Con}(A)$, for some $A \in \mathcal{K}$, (since $\text{SPCon}(\mathcal{K}) = \text{SCon}(\mathcal{K})$).
An Application

Theorem

Let \( K \) is a variety with congruence variety \( \mathcal{V} \). Suppose \( \mathcal{V} \neq \mathcal{L} \) and \( M_3 \in \mathcal{V} \). Then \( M_3 \) is projective in \( \mathcal{V} \).

Proof.

- \( F_{\mathcal{V}}(3) \) is a sublattice of \( \text{Con} (A) \), for some \( A \in K \),
- Let \( \alpha, \beta, \) and \( \gamma \) be the free generators.
An Application

Theorem

Let $\mathcal{K}$ is a variety with congruence variety $\mathcal{V}$. Suppose $\mathcal{V} \neq \mathcal{L}$ and $M_3 \in \mathcal{V}$. Then $M_3$ is projective in $\mathcal{V}$.

Proof.

- $F_{\mathcal{V}}(3)$ is a sublattice of $\text{Con}(A)$, for some $A \in \mathcal{K}$,
- Let $\alpha$, $\beta$, and $\gamma$ be the free generators.
- By the above theorem, $\beta^m = \beta^{m+1}$ and $\gamma^m = \gamma^{m+1}$, for some $m$, for $M_3$ is projective in $\mathcal{V}$. 
An Application

Theorem

Let $\mathcal{K}$ be a variety with congruence variety $\mathcal{V}$. Suppose $\mathcal{V} \neq \mathcal{L}$ and $M_3 \in \mathcal{V}$. Then $M_3$ is projective in $\mathcal{V}$.

Proof.

- $F_\mathcal{V}(3)$ is a sublattice of $\text{Con}(A)$, for some $A \in \mathcal{K}$,
- Let $\alpha, \beta$, and $\gamma$ be the free generators.
- By the above theorem, $\beta^m = \beta^{m+1}$ and $\gamma^m = \gamma^{m+1}$, for some $m$,
- which implies $\alpha \vee \beta^m = \alpha \vee \gamma^m$. 

An Application

Theorem

Let $\mathcal{K}$ is a variety with congruence variety $\mathcal{V}$. Suppose $\mathcal{V} \neq \mathcal{L}$ and $M_3 \in \mathcal{V}$. Then $M_3$ is projective in $\mathcal{V}$.

Proof.

- $F_\mathcal{V}(3)$ is a sublattice of $\text{Con}(A)$, for some $A \in \mathcal{K}$,
- Let $\alpha$, $\beta$, and $\gamma$ be the free generators.
- By the above theorem, $\beta^m = \beta^{m+1}$ and $\gamma^m = \gamma^{m+1}$, for some $m$,
- which implies $\alpha \lor \beta^m = \alpha \lor \gamma^m$.
- So the interval from $\alpha \lor (\beta^m \land \gamma^m)$ to $\alpha \lor \beta^m$ is abelian.
An Application

Theorem

Let $\mathcal{K}$ is a variety with congruence variety $\mathcal{V}$. Suppose $\mathcal{V} \neq \mathcal{L}$ and $M_3 \in \mathcal{V}$. Then $M_3$ is projective in $\mathcal{V}$.

Proof.

- $F_\mathcal{V}(3)$ is a sublattice of $\text{Con}(A)$, for some $A \in \mathcal{K}$,
- Let $\alpha$, $\beta$, and $\gamma$ be the free generators.
- By the above theorem, $\beta^m = \beta^{m+1}$ and $\gamma^m = \gamma^{m+1}$, for some $m$,
- which implies $\alpha \vee \beta^m = \alpha \vee \gamma^m$.
- So the interval from $\alpha \vee (\beta^m \wedge \gamma^m)$ to $\alpha \vee \beta^m$ is abelian.
- A picture:
An Application

[Diagram]

\[ \alpha + \beta^m = \alpha + \gamma^m \]

\[ \beta^m \]

\[ \alpha + \beta^m \gamma^m \]

\[ \gamma^m \]

\[ \beta' = \beta^m(\alpha + \beta^m \gamma^m) \]

\[ \gamma' \]
If $f(\alpha) = a$, $f(\beta) = b$ and $f(\gamma) = c$ is the homomorphism onto $\mathbb{M}_3$, then $f(\beta^m) = b$ and $f(\beta') = 0$. 

$$\alpha + \beta^m = \alpha + \gamma^m$$

$$\beta' = \beta^m(\alpha + \beta^m \gamma^m)$$
If \( f(\alpha) = a, f(\beta) = b \) and \( f(\gamma) = c \) is the homomorphism onto \( M_3 \), then \( f(\beta^m) = b \) and \( f(\beta') = 0 \). Also the interval \( I[\beta', \beta^m] \) is abelian.
In the sublattice generated by \( \beta^m, \beta', \gamma^m, \gamma' \)

\[
\beta'' := \beta^m + \gamma' \quad \gamma'' := \gamma^m + \beta'
\]

\[
\beta^m \quad \gamma^m \\
\beta' \quad \gamma'
\]
In the sublattice generated by $\beta^m, \beta', \gamma^m, \gamma'$

$\beta'' := \beta^m + \gamma'$

$\gamma'' := \gamma^m + \beta'$

If $\beta'' = \beta^m + \gamma'$ and $\gamma'' = \gamma^m + \beta'$, then the interval $\langle \beta'', \gamma'', \beta'' + \gamma'' \rangle$ is solvable and hence modular.
In the sublattice generated by $\beta^m, \beta', \gamma^m, \gamma'$

- If $\beta'' = \beta^m + \gamma'$ and $\gamma'' = \gamma^m + \beta'$, then the interval $I[\beta''\gamma'', \beta'' + \gamma'']$ is solvable and hence modular.
- Letting $\alpha'' = \alpha(\beta'' + \gamma'') + \beta''\gamma''$ we get preimages of $a$, $b$ and $c$ in a modular interval.
In the sublattice generated by $\beta^m, \beta', \gamma^m, \gamma'$

If $\beta'' = \beta^m + \gamma'$ and $\gamma'' = \gamma^m + \beta'$, then the interval $I[\beta''\gamma'', \beta'' + \gamma'']$ is solvable and hence modular.

Letting $\alpha'' = \alpha(\beta'' + \gamma'') + \beta''\gamma''$ we get preimages of $a, b$ and $c$ in a modular interval.

The result follows from the projectivity of $M_3$ in modular lattices.
Corollary

If $L$ is a nonargesian projective plane, then $L$ is not in any proper congruence variety.
A Corollary

Corollary

If $L$ is a nonargesian projective plane, then $L$ is not in any proper congruence variety.

Recall the proof that lattices of permuting equivalence relations are arguesian; they satisfy

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1)$$  \hspace{1cm} (1)

where $c_0 = (a_1 + a_2)(b_1 + b_2)$, etc.
Corollary

If $L$ is a nonargesian projective plane, then $L$ is not in any proper congruence variety.

- Recall the proof that lattices of permuting equivalence relations are arguesian; they satisfy
  \[
  (a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1) \tag{1}
  \]
  where $c_0 = (a_1 + a_2)(b_1 + b_2)$, etc.
- If these are (permuting) equivalence relations and $(x, y)$ is in it, then
A Corollary

Theorem

If \(a_i\) and \(b_i\) are elements of a lattice of equivalence relations and \(a_i\) permutes with \(b_i\), \(i = 0, 1\) and 2, then

\[
(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1)
\]
A Corollary

\[(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1)\]
A Corollary

Theorem

If $a_i$ and $b_i$ are elements of a lattice of equivalence relations and $a_i$ permutes with $b_i$, $i = 0, 1, 2$, then

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1)$$

holds.

$$c_2' = c_2(c_0 + c_1)$$
A Corollary

Theorem

If \( a_i \) and \( b_i \) are elements of a lattice of equivalence relations and \( a_i \) permutes with \( b_i \), \( i = 0, 1 \) and \( 2 \), then

\[
(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1)
\]

holds.

\( c'_2 = c_2(c_0 + c_1) \)
A Corollary

If \(a_i\) and \(b_i\) are elements of a lattice of equivalence relations and \(a_i\) permutes with \(b_i\), \(i = 0, 1\), and 2, then

\[
(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1)
\]

holds.
Assume $L \in HSCon(\mathcal{K})$, where $\mathcal{K}$ satisfies a congruence identity.
The Proof

- Assume \( L \in HSC_{\text{Con}}(\mathcal{K}) \), where \( \mathcal{K} \) satisfies a congruence identity.
- Let \( f : L' \rightarrow L \), where \( L' \leq \text{Con}(A) \).
The Proof

- Assume $L \in HSCon(\mathcal{K})$, where $\mathcal{K}$ satisfies a congruence identity.
- Let $f : L' \to L$, where $L' \leq Con(A)$.
- Let $a_i, b_i, i = 0, 1, 2$ be points (atoms) of $L$ witnessing the failure of the arguesian identity.
Assume $L \in HSCon (\mathcal{K})$, where $\mathcal{K}$ satisfies a congruence identity.

Let $f : L' \to L$, where $L' \leq \text{Con} (A)$.

Let $a_i, b_i, i = 0, 1, 2$ be points (atoms) of $L$ witnessing the failure of the arguesian identity.

And let $d_i \in L$ be a point on the line $a_i + b_i$ so $a_i, b_i, d_i$ are the atoms of an $\mathcal{M}_3$. 
The Proof

- Assume $L \in HSCon(\mathcal{K})$, where $\mathcal{K}$ satisfies a congruence identity.
- Let $f : L' \rightarrow L$, where $L' \leq \text{Con}(A)$.
- Let $a_i, b_i, i = 0, 1, 2$ be points (atoms) of $L$ witnessing the failure of the arguesian identity.
- And let $d_i \in L$ be a point on the line $a_i + b_i$ so $a_i, b_i, and d_i$ are the atoms of an $M_3$.
- By the projectivity of $M_3$, we can find inverse images $\alpha_i, \beta_i$ and $\delta_i$ forming an $M_3$. 
The Proof

- Assume $L \in HSC_{\text{Con}} (\mathcal{K})$, where $\mathcal{K}$ satisfies a congruence identity.
- Let $f : L' \rightarrow L$, where $L' \leq \text{Con} (A)$.
- Let $a_i, b_i, i = 0, 1, 2$ be points (atoms) of $L$ witnessing the failure of the arguesian identity.
- And let $d_i \in L$ be a point on the line $a_i + b_i$ so $a_i, b_i,$ and $d_i$ are the atoms of an $\mathbf{M}_3$.
- By the projectivity of $\mathbf{M}_3$, we can find inverse images $\alpha_i, \beta_i$ and $\delta_i$ forming an $\mathbf{M}_3$.
- Thus $\alpha_i$ and $\beta_i$ permute. So
The Proof

- Assume $L \in HSCon(\mathcal{K})$, where $\mathcal{K}$ satisfies a congruence identity.
- Let $f : L' \to L$, where $L' \leq \text{Con}(A)$.
- Let $a_i, b_i, i = 0, 1, 2$ be points (atoms) of $L$ witnessing the failure of the arguesian identity.
- And let $d_i \in L$ be a point on the line $a_i + b_i$ so $a_i, b_i, \text{and } d_i$ are the atoms of an $M_3$.
- By the projectivity of $M_3$, we can find inverse images $\alpha_i, \beta_i$ and $\delta_i$ forming an $M_3$.
- Thus $\alpha_i$ and $\beta_i$ permute. So

$$
(\alpha_0 + \beta_0)(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) \leq \alpha_0 + \beta_1 + \gamma_2(\gamma_0 + \gamma_1).
$$
Assume \( L \in HSC_{\text{Con}}(\mathcal{K}) \), where \( \mathcal{K} \) satisfies a congruence identity.

Let \( f : L' \rightarrow L \), where \( L' \leq \text{Con}(A) \).

Let \( a_i, b_i, i = 0, 1, 2 \) be points (atoms) of \( L \) witnessing the failure of the arguesian identity.

And let \( d_i \in L \) be a point on the line \( a_i + b_i \) so \( a_i, b_i, \) and \( d_i \) are the atoms of an \( M_3 \).

By the projectivity of \( M_3 \), we can find inverse images \( \alpha_i, \beta_i \) and \( \delta_i \) forming an \( M_3 \).

Thus \( \alpha_i \) and \( \beta_i \) permute. So

\[
(\alpha_0 + \beta_0)(\alpha_1 + \beta_1)(\alpha_2 + \beta_2) \leq \alpha_0 + \beta_1 + \gamma_2(\gamma_0 + \gamma_1).
\]

Applying \( f \) shows

\[
(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1),
\]

a contradiction.
Mark Haiman produced stronger and stronger identities \((D_n)\) which hold in lattices of permuting equivalence relations,
Mark Haiman produced stronger and stronger identities \((D_n)\) which hold in lattices of permuting equivalence relations, and lattices \(H_n\) showing these identities really are distinct.
Mark Haiman produced stronger and stronger identities \((D_n)\) which hold in lattices of permuting equivalence relations, and lattices \(H_n\) showing these identities really are distinct. \(H_n\) is generated by \(n + 3\) and no fewer elements.
Mark Haiman produced stronger and stronger identities \((D_n)\) which hold in lattices of permuting equivalence relations, and lattices \(H_n\) showing these identities really are distinct. 

\(H_n\) is generated by \(n + 3\) and no fewer elements.

Every proper sublattice of \(H_n\) can be embedded into the lattice of subspaces of a vector space.
Mark Haiman produced stronger and stronger identities \((D_n)\) which hold in lattices of permuting equivalence relations, and lattices \(H_n\) showing these identities really are distinct. 

\(H_n\) is generated by \(n + 3\) and no fewer elements. 

Every proper sublattice of \(H_n\) can be embedded into the lattice of subspaces of a vector space. 

A proof similar to the one above shows
Mark Haiman produced stronger and stronger identities \((D_n)\) which hold in lattices of permuting equivalence relations, and lattices \(H_n\) showing these identities really are distinct. \(H_n\) is generated by \(n + 3\) and no fewer elements. Every proper sublattice of \(H_n\) can be embedded into the lattice of subspaces of a vector space.

A proof similar to the one above shows

**Theorem**

Haiman’s lattices, \(H_n\), lie in no proper congruence variety.
Finitely Based Congruence Varieties

Theorem (with P. Lipparini)

If \( \mathcal{K} \) is a variety whose congruence variety is finitely based and is not \( \mathcal{L} \), then \( \mathcal{K} \) is congruence semidistributive.
Theorem (with P. Lipparini)

If $\mathcal{K}$ is a variety whose congruence variety is finitely based and is not $\mathcal{L}$, then $\mathcal{K}$ is congruence semidistributive.

Proof.

- If $\mathcal{K}$ is not semidistributive, $\mathcal{M}_p \subseteq \text{VCon}(\mathcal{K})$, for some $p$. 
Theorem (with P. Lipparini)

*If* \( \mathcal{K} \) *is a variety whose congruence variety is finitely based and is not* \( \mathcal{L} \), *then* \( \mathcal{K} \) *is congruence semidistributive.*

Proof.

- If \( \mathcal{K} \) is not semidistributive, \( \mathcal{M}_p \subseteq \text{VCon} (\mathcal{K}) \), for some \( p \).
- \( H_n \notin \text{VCon} (\mathcal{K}) \),
Theorem (with P. Lipparini)

If $\mathcal{K}$ is a variety whose congruence variety is finitely based and is not $\mathcal{L}$, then $\mathcal{K}$ is congruence semidistributive.

Proof.

- If $\mathcal{K}$ is not semidistributive, $\mathcal{M}_p \subseteq \text{VCon}(\mathcal{K})$, for some $p$.
- $H_n \notin \text{VCon}(\mathcal{K})$,
- so $\text{VCon}(\mathcal{K})$ is not finitely based.