BRANCHES OF $z^c$

1. Introduction

Definition 1. For complex numbers $z$ and $c$, with $z \neq 0$, we set

$$z^c = e^{c \log(z)}$$

where $\log(z) = \ln|z| + i(\text{Arg}(z) + 2\pi n)$ for any integer $n \in \mathbb{Z}$. We say that $f$ is a branch of $z^c$ if there is some open set $U \subset \mathbb{C}$ with

(i) $0 \not\in U$ and $U$ is connected
(ii) $f : U \to \mathbb{C}$
(iii) $f$ is continuous on $U$
(iv) For each $z \in U$, there is some integer $n_z$ such that

$$f(z) = \exp\{c \cdot [\ln|z| + i\text{Arg}(z) + 2\pi in_z]\}$$

That is, $f$ is a continuous version of $z^c$, defined on some open, connected set $U$ with $0 \not\in U$.

Lemma 1. Let $z_0 \neq 0$, written as $x_0 + iy_0$ with $x_0$ and $y_0$ real. There is a branch of $\log(z)$ for $z$ such that $|z - z_0| < \max\{|x_0|, |y_0|\}$.

Note: The book does this slightly differently, with a different argument, in the middle of page 41, for the disk $|z - z_0| < |z_0|$.

Proof. We proved this in class, with four different formulas for a branch of $\text{arg}(w)$.

Let $z = x + iy$, with $x$ and $y$ real. Set $r = |z|$. We can set

$$\alpha(z) = \begin{cases} 
\arctan(y/x) & \text{if } x > 0 \\
\pi + \arctan(y/x) & \text{if } x < 0 \\
\arccos(x/r) & \text{if } y > 0 \\
-\arccos(x/r) & \text{if } y < 0 
\end{cases}$$

Each of these cases provides a continuous version of $\text{arg}$ on an open a half-plane with either the $x$ or the $y$ axis as a boundary. The disc $D = \{ z \in \mathbb{C} : |z - z_0| < \max\{|x_0|, |y_0|\} \}$ is a subset of one of these half-planes. To see this latter fact, explore cases.

- Suppose $|x_0| \leq |y_0|$. Because $z_0 \neq 0$, we have $|y_0| > 0$. Let $|z - z_0| < |y_0|$. Then

$$|y - y_0| \leq |z - z_0| < |y_0|$$

and thus $y_0 - |y_0| < y < y_0 + |y_0|$$

If $y_0 > 0$, then $y > y_0 - |y_0| > 0$. If $y_0 < 0$, then $y < y_0 + |y_0| = 0$.

- Suppose $|y_0| < |x_0|$. Because $z_0 \neq 0$, we have $|x_0| > 0$. Let $|z - z_0| < |x_0|$. Then

$$|x - x_0| \leq |z - z_0| < |x_0|$$

and thus $x_0 - |x_0| < x < x_0 + |x_0|$$

If $x_0 > 0$, then $x > x_0 - |x_0| > 0$. If $x_0 < 0$, then $x < x_0 + |x_0| = 0$. 

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Once we have a continuous version of \( \arg(z) \) on \( D \), we can create from it a branch of \( \log(z) \):
\[
\ell(z) = \ln |z| + i \alpha(z)
\]
\[\Box\]

**Proposition 1.** Let \( f \) be a branch of \( z^c \) on an open set \( U \). For each \( z_0 \in U \) there is some \( \delta > 0 \) and at least one choice of \( \ell \) as in Lemma 1 so that, with \( D = \{ z \in \mathbb{C} : |z - z_0| < \delta \} \),
- \( D \subset U \) and \( D \) is a subset of the domain of \( \ell \)
- There is a constant \( K = e^{2\pi icm} \) for some integer \( m \) such that, for all \( z \in D \),
\[
f(z) = K \cdot e^{\ell f(z)}
\]
- Therefore, \( f \) is holomorphic on \( D \) and
\[
f'(z) = K e^{\ell f(z)} \cdot c \cdot \frac{1}{z} = \frac{cf(z)}{z}
\]
- The last formula holds for an open neighborhood of every \( z_0 \in U \). Hence it holds on all of \( U \).
- \( f' \) is continuous on \( U \), because \( c \) is a constant, \( f \) is continuous by the definition of a branch, and \( 1/z \) is continuous on \( U \) because \( 0 \not\in U \).
- We can write \( 1/z = e^{-\ell(z)} \) because \( e^{-w} = 1/(e^w) \). So we have another formula for \( f' \) on \( D \):
\[
f'(z) = K e^{\ell f(z)} \cdot c \cdot e^{-\ell(z)} = K ce^{(c-1)\ell(z)}
\]
- In particular, since \( K = e^{2\pi icm} \) for some integer \( m \), for \( z \in D \)
\[
f'(z) = ce^{(c-1)\ell(z)+c(2\pi im)} = ce^{(c-1)(\ell(z)+2\pi im)+2\pi im}
\]
\[
= ce^{(c-1)(\ell(z)+2\pi im)}
\]

With \( m \) fixed and an integer, \( \ell(z) + 2\pi im \) is another branch of \( \log \) on \( D \) and that makes \( f' \) a branch of \( cz^{c-1} \) on \( D \).
- Because the preceding item holds for a neighborhood \( D \) at every \( z_0 \) in \( U \), and \( f' \) is continuous on \( U \), \( f' \) is a branch of \( cz^{c-1} \) on \( U \).

**Proof.** Let \( z_0 \in U \). Because \( 0 \not\in U \), we have \( z_0 \neq 0 \) and therefore \( \max\{|x_0|,|y_0|\} > 0 \).

Because \( U \) is open, there is some \( \delta_1 \) such that \( |z - z_0| < \delta_1 \) implies \( z \in U \). Let \( \delta \) be the smaller of \( \delta_1 \) and \( \max\{|x_0|,|y_0|\} \).

Set \( D = \{ z \in \mathbb{C} : |z - z_0| < \delta \} \). By our choice of \( \delta \leq \delta_1 \), we have \( D \subset U \). By our choice of \( \delta \leq \max\{|x_0|,|y_0|\} \), we have \( D \) a subset of one of the four half-planes in Lemma 1, on which we may choose a branch \( \ell(z) \) of \( \log \).

Let \( g(z) = \exp(c\ell(z)) \) for \( z \in D \). It is a branch of \( z^c \) and clearly never zero. Set
\[
h(z) = \frac{f(z)}{g(z)}
\]
Note that \( h \) is continuous on \( D \), as the quotient of two continuous functions.

Because \( f(z) \) and \( g(z) \) are versions of \( z^c \), there are integers \( j_k \) and \( k_k \) such that
\[
f(z) = \exp\{ c \cdot [\ln|z| + i\text{Arg}(z) + 2\pi ij_k] \}
\]
and
\[ g(z) = \exp\{ c \cdot [\ln |z| + i\text{Arg}(z) + 2\pi i k_z] \} \]

Let \( m_z = j_z - k_z \). Then
\[ h(z) = \exp(2\pi i cm_z) \]

Suppose \( \Re(h(z_1)) \neq \Re(h(z_0)) \) for some \( z_1 \in D \). On the line segment \([z_0, z_1]\), which is a subset of \( D \), \( \Re(h(z)) \) is real-valued and continuous, because it is the composition of continuous functions (\( z \mapsto \Re(z) \) is continuous from \( \mathbb{C} \) to \( \mathbb{R} \)). The segment \([z_0, z_1]\) is a connected set. So the range of \( \Re(h) \) on \([z_0, z_1]\) has to be connected and hence an interval (the only connected subsets of \( \mathbb{R} \) are intervals).

The interval has positive length because \( \Re(h(z_1)) \neq \Re(h(z_0)) \).

That makes the range of \( h(z) \) on \( D \) uncountable. However there are only countably many choices for \( m_z \) (these are all integers). This is a contradiction. So the real of \( h(z) \) has to be constant on \( D \).

Suppose \( \Im(h(z_1)) \neq \Im(h(z_0)) \) for some \( z_1 \in D \). On the line segment \([z_0, z_1]\), which is a subset of \( D \), \( \Im(h(z)) \) is real-valued and continuous, because it is the composition of continuous functions (\( z \mapsto \Im(z) \) is continuous from \( \mathbb{C} \) to \( \mathbb{R} \)). The segment \([z_0, z_1]\) is a connected set. So the range of \( \Im(h) \) on \([z_0, z_1]\) has to be connected and hence an interval (the only connected subsets of \( \mathbb{R} \) are intervals).

The interval has positive length because \( \Im(h(z_1)) \neq \Im(h(z_0)) \).

That makes the range of \( h(z) \) on \( D \) uncountable. However there are only countably many choices for \( m_z \) (these are all integers). This is a contradiction. So the real of \( h(z) \) has to be constant on \( D \).

\(\square\)