\( \phi \) is **piecewise continuous** on \([a, b]\) if and only if

- \( a \leq b \) in \( \mathbb{R} \) and \( \phi : [a, b] \rightarrow \mathbb{C} \)
- There is a finite set \( S \subset [a, b] \) such that, for all \( t \in [a, b] \setminus S \), \( \phi \) is continuous at \( t \):

\[
\phi(t) = \lim_{u \to t} \phi(u) \quad \forall u \in [a, b]
\]

- For all \( t \in [a, b) \), \( \phi \) has a finite right limit at \( t \).
- For all \( t \in (a, b] \), \( \phi \) has a finite left limit at \( t \).
Lemma: \( \phi \) is piecewise continuous on \([a, b]\) if and only if \( \Re(\phi) \) and \( \Im(\phi) \) are piecewise continuous on \([a, b]\).

**Theorem from Advanced Calculus:** \( f : [a, b] \rightarrow \mathbb{R} \) is Riemann integrable if and only if \( f \) is bounded and the set of discontinuities of \( f \) has outer measure 0.

**Special Problem (Chi):** Let \( f \) be real-valued and piecewise continuous on \([a, b]\). Then

1. \( f \) is bounded
2. \( f \) is Riemann integrable
Special Problem (Kodgis): Assume that
- $\phi$ is piecewise continuous on $[a, b]$.
- $g : E \to \mathbb{C}$ with $E \subset \mathbb{C}$
- $g$ is continuous on an open set $U \subset E$
- There is a closed set $F$ such that range$(\phi) \subset F \subset U$.

Then $g \circ \phi$ is piecewise continuous on $[a, b]$.

Example: Note that the absolute value function is continuous on all of $\mathbb{C}$. Consequently, if $\phi$ is piecewise continuous, so is $|\phi|$. This is used implicitly in Section 3 of Chapter 6 on page 66.
Special Problem (Chevalier): Suppose that $\phi$ is piecewise continuous on $[a, b]$. Suppose that $\tau : [c, d] \to [a, b]$ is one-to-one and continuous. Then $\phi \circ \tau$ is piecewise continuous on $[c, d]$.

**Hint:** You can prove that $\tau$ is strictly increasing or strictly decreasing. That leads to two cases.
Def’n: If $\phi$ is piecewise continuous on $[a, b]$ we define $\int_a^b \phi(t) \, dt$ by

$$\int_a^b \phi(t) \, dt := \int_a^b \Re(\phi(t)) \, dt + i \int_a^b \Im(\phi(t)) \, dt$$

Special Problem (Reckwerdt): Let $\phi$ and $\lambda$ be piecewise continuous on $[a, b]$. Then $\phi + \lambda$ is piecewise continuous on $[a, b]$, and

$$\int_a^b (\phi + \lambda)(t) \, dt = \int_a^b \phi(t) \, dt + \int_a^b \lambda(t) \, dt$$
Special Problem (Feliciano): If $\phi$ is piecewise continuous on $[a, b]$ and $c$ is a complex number, then $c\phi$ is piecewise continuous on $[a, b]$ and

$$\int_a^b (c\phi)(t) \, dt = c \int_a^b \phi(t) \, dt$$

Theorem, page 65: If $\phi_1$ and $\phi_2$ are piecewise continuous on $[a, b]$, and $c_1$ and $c_2$ are complex numbers, then $c_1\phi_1 + c_2\phi_2$ is piecewise continuous on $[a, b]$ and

$$\int_a^b [c_1\phi_1 + c_2\phi_2](t) \, dt = c_1 \int_a^b \phi_1(t) \, dt + c_2 \int_a^b \phi_2(t) \, dt$$
Special Problem (Rader) Suppose that $\phi$ is piecewise continuous on $[a, b]$. Suppose that $[c, d] \subset [a, b]$ with $c < d$. Then

$$\int_{c}^{d} \phi(t) \, dt = \lim_{\delta \downarrow 0} \int_{c+\delta}^{d-\delta} \phi(t) \, dt$$

Proof Hints: You may use for free that, when $0 < \delta < (d - c)/2$,

$$\int_{c}^{c+\delta} \phi(t) \, dt + \int_{c+\delta}^{d-\delta} \phi(t) \, dt + \int_{d-\delta}^{d} \phi(t) \, dt = \int_{c}^{d} \phi(t) \, dt$$

Somehow use the bounded-ness of $\phi$ to finish the argument.
The previous problem is valid more generally. It remains valid if

- $\Re(\phi)$ and $\Im(\phi)$ are Riemann integrable on $[a, b]$
- $\Re(\phi)$ and $\Im(\phi)$ are Lebesgue measurable on $[a, b]$ with

\[
\int_a^b |\Re(\phi(t))| \, dt < \infty
\]

and

\[
\int_a^b |\Im(\phi(t))| \, dt < \infty
\]
Special Problem (Hedley): Let $\phi$ and $\psi$ be piecewise continuous on $[a, b]$. Suppose there is a finite set $W$ such that, for all $t \in [a, b] \setminus W$,

$$\phi(t) = \psi(t)$$

Then

$$\int_a^b \phi(t) \, dt = \int_a^b \psi(t) \, dt$$
Lemma: Let $\phi$ and $\psi$ be piecewise continuous on $[a, b]$. Then $\phi \cdot \psi$ is piecewise continuous on $[a, b]$.

Lemma: Let $\phi : [a, b] \to \mathbb{C}$. Let $a < c < b$. Set $\lambda$ equal to the restriction of $\phi$ to $[a, c]$ and $\psi$ equal to the restriction of $\phi$ to $[c, b]$. Then

- $\phi$ is piecewise continuous on $[a, b]$ if and only if $\lambda$ is piecewise continuous on $[a, c]$ and $\psi$ is piecewise continuous on $[c, b]$.
- If $\phi$ is piecewise continuous on $[a, b]$, then

$$\int_a^b \phi(t) \, dt = \int_a^c \lambda(t) \, dt + \int_c^b \psi(t) \, dt$$
Differentiating $\phi : [a, b] \to \mathbb{C}$

**Def’n:** Let $a < b$ and $\phi : [a, b] \to \mathbb{C}$. We say that $\phi$ is differentiable at $t_0$ if and only if $t_0 \in [a, b]$ and there is a complex number $L$ such that

$$L = \lim_{\substack{u \to t_0 \\ u \in [a, b]}} \frac{\phi(u) - \phi(t_0)}{u - t_0}$$

$L$ is necessarily unique, and we call it $\phi'(t_0)$.

**Special Problem (Billington):** Let $a < b$ and $\phi : [a, b] \to \mathbb{C}$. Then

1. $\phi$ is differentiable at $t_0$ if and only if $\Re(\phi)$ and $\Im(\phi)$ are differentiable at $t_0$.
2. If $\phi$ is differentiable at $t_0$ then

$$\phi'(t_0) = [\Re(\phi)]'(t_0) + i \cdot [\Im(\phi)]'(t_0)$$

Note that this problem says that, when $\phi'(t)$ exists, $[\Re(\phi)]'(t) = \Re[\phi'(t)]$ and $[\Im(\phi)]'(t) = \Im[\phi'(t)]$. 

Ramsey Complex Integration
Def’n: Let $a < b$ and $\phi : [a, b] \rightarrow \mathbb{C}$. $\phi$ is piecewise $C^1$ if and only if

- $\phi$ is continuous on $[a, b]$
- There is a finite set $S$ such that, for all $t \in [a, b] \setminus S$ $\phi$ is differentiable at $t$.
- For all $t \in [a, b]$, if $\phi$ is differentiable at $t$ then $\phi'$ is continuous at $t$.
- For all $t \in [a, b)$, $\phi'$ has a finite right limit at $t$.
- For all $t \in (a, b]$, $\phi'$ has a finite left limit at $t$. 


Special Problem (Holmes): Let $a < b$, $\phi : [a, b] \to \mathbb{C}$, and $\phi$ piecewise $C^1$. Suppose that $g : E \to \mathbb{C}$, with $E \subset \mathbb{C}$. Let $F \subset G \subset E$ such that

- $F$ is closed and $G$ is open
- The range of $\phi$ is a subset of $F$
- $g$ is holomorphic on $G$
- $g$ is $C^1$ on $G$ (meaning, if $g = u + iv$ with $u$ and $v$ real, $u_x(z)$, $u_y(z)$, $v_x(z)$ and $v_y(z)$ are continuous at every $z \in G$).

Then $g \circ \phi$ is piecewise $C^1$. 
Special Problem (Thompson): Let $a < b$, $\phi : [a, b] \to \mathbb{C}$, and $\phi$ piecewise $C^1$. Let $\tau : [c, d] \to [a, b]$ be one-to-one with a continuous derivative ($C^1$). Then $\phi \circ \tau$ is piecewise $C^1$.

Hint: Prove that $\tau$ must be strictly decreasing or strictly increasing. That leads to two cases.
Lemma: Let \( a < b, \phi : [a, b] \rightarrow \mathbb{C} \). Then \( \phi \) is piecewise \( C^1 \) if and only if \( \Re(\phi) \) and \( \Im(\phi) \) are piecewise \( C^1 \).

Fundamental Theorem of Calculus: Let \( a < b, \phi : [a, b] \rightarrow \mathbb{C} \). Suppose that \( \phi \) is piecewise \( C^1 \). Then \( S \) is finite, where \( S \) is the set of \( t \in [a, b] \) such that \( \phi \) is not differentiable at \( t \). Let \( h : [a, b] \rightarrow \mathbb{C} \) satisfy

- \( h(t) = \phi'(t) \) for \( t \in [a, b] \setminus S \).
- \( h \) on \( S \) can be an arbitrary, independent assignment of complex numbers.

Then \( h \) is Riemann integrable and

\[
\int_{a}^{b} h(t) \, dt = \phi(b) - \phi(a)
\]
Claim 1: $h$ is continuous at $t \in [a, b] \setminus S$.

Proof: Let $t \notin S$. Because $S$ is finite, there is some $\delta_0 > 0$ such that

$$|u - t| < \delta_0 \implies u \notin S$$

Thus, if $|u - t| < \delta_0$ and $u \in [a, b]$, we have $h(u) = \phi'(u)$.

- Because $\phi'(t)$ exists (since $t \notin S$), $\phi'$ is continuous at $t$ by hypothesis.
- Because $\phi'$ and $h$ agree on $[a, b] \cap (t - \delta_0, t + \delta_0)$ we have $h$ continuous at $t$. 
Claim 2: For all $t \in [a, b)$, $h$ has a finite right limit at $t$.

Proof: Let $t \in [a, b)$.

- If $t \notin S$, then $h$ is continuous at $t$. Because $t \neq b$, this implies that the right limit of $h$ at $t$ is $h(t)$, which is finite.

- Suppose that $t \in S$. Because $S$ is finite, there is some $\delta_0 > 0$ such that $(t - \delta_0, t + \delta_0) \cap S = \{t\}$. Hence

  \[ u \in (t, \min\{t + \delta_0, b\}) \quad \Rightarrow \quad h(u) = \phi'(u) \]

- Note that both $t + \delta_0$ and $b$ are strictly bigger than $t$. Thus $h$ and $\phi$ agree on an open interval of positive length with left end $t$.

- Because $\phi'$ has a finite right limit at $t$, so does $h$. 
Claim 3: For all $t \in (a, b]$, $h$ has a finite left limit at $t$.

Proof: Let $t \in [a, b)$.

- If $t \notin S$, then $h$ is continuous at $t$. Because $t \neq a$, this implies that the left limit of $h$ at $t$ is $h(t)$, which is finite.
- Suppose that $t \in S$. Because $S$ is finite, there is some $\delta_0 > 0$ such that $(t - \delta_0, t + \delta_0) \cap S = \{t\}$. Hence
  \begin{equation*}
  u \in (\min\{a, t - \delta_0\}, t) \implies h(u) = \phi'(u)
  \end{equation*}
  Note that both $t - \delta_0$ and $a$ are strictly less than $t$. Thus $h$ and $\phi$ agree on an open interval of positive length with right end $t$.
- Because $\phi'$ has a finite left limit at $t$, so does $h$. 
By Claims 1, 2 and 3, $h$ is piecewise continuous on $[a, b]$ and thus Riemann integrable.

**Claim 4:** Let $c < d$ in $[a, b]$ with $(c, d) \cap S = \emptyset$. Then

$$\int_{c}^{d} h(t) \, dt = \phi(d) - \phi(c)$$

**Proof:** Consider $\delta < (d - c)/2$. Then $J = [c + \delta, d - \delta] \subset (c, d)$.

- Hence no point of $S$ is in $J$ and thus $h(u) = \phi'(u)$ for $u \in J$.
- Note that $\phi'$ is continuous on $J$ because it is continuous on the open interval $(c, d)$ containing $J$.
- Because $\phi' = [\Re(\phi)]' + i[\Im(\phi)]'$, that makes both $[\Re(\phi)]'$ and $[\Im(\phi)]'$ continuous on $J$. 
By the Fundamental Theorem of Calculus, applied to the real and imaginary parts of $\phi'$, we have

\[
\int_{c+\delta}^{d-\delta} h(t) \, dt = \int_{c+\delta}^{d-\delta} \phi'(t) \, dt
\]

\[
= \int_{c+\delta}^{d-\delta} \Re[\phi'(t)] \, dt + i \int_{c+\delta}^{d-\delta} \Im[\phi'(t)] \, dt
\]

\[
= \int_{c+\delta}^{d-\delta} \left( \Re(\phi)'(t) \right) \, dt + i \int_{c+\delta}^{d-\delta} \left( \Im(\phi)'(t) \right) \, dt
\]

\[
= \left\{ \left[ \Re(\phi) \right](d - \delta) - \left[ \Re(\phi) \right](c + \delta) \right\}
\]

\[
+ i \left\{ \left[ \Im(\phi) \right](d - \delta) - \left[ \Im(\phi) \right](c + \delta) \right\}
\]

\[
= \phi(d - \delta) - \phi(c + \delta)
\]
By Rader’s Special Problem,

\[ \int_{c}^{d} h(t) \, dt = \lim_{\delta \downarrow 0} \int_{c+\delta}^{d-\delta} h(t) \, dt \]

Therefore

\[ \int_{c}^{d} h(t) \, dt = \lim_{\delta \downarrow 0} \{ \phi(d - \delta) - \phi(c + \delta) \} \]

Because \( \phi \) is \( C^1 \), \( \phi \) is continuous. Therefore the left limit at \( d \) of \( \phi \) is \( \phi(d) \) and the right limit at \( c \) of \( \phi \) is \( \phi(c) \). Hence, since difference in \( \mathbb{C} \) preserves limits,

\[ \int_{c}^{d} h(t) \, dt = \phi(d) - \phi(c) \]
Suppose $S \cap (a, b) = \emptyset$. Apply Claim 4 with $[c, d] = [a, b]$ to conclude that
\[
\int_a^b h(t) \, dt = \phi(b) - \phi(a)
\]

Suppose that $S \cap (a, b) \neq \emptyset$. Let $\{s_j\}_{j=1}^n$ enumerate $S \cap (a, b)$, with $s_j < s_{j+1}$ for $1 \leq j \leq n - 1$.

Let $s_0 = a$ and $s_{n+1} = b$.

For $0 \leq j \leq n$, $(s_j, s_{j+1}) \cap S = \emptyset$.

By Claim 4,
\[
\int_{s_j}^{s_{j+1}} h(t) \, dt = \phi(s_{j+1}) - \phi(s_j)
\]
Hence

\[ \int_{a}^{b} h(t) \, dt = \sum_{j=0}^{n} \int_{s_j}^{s_{j+1}} h(t) \, dt \]

\[ = \sum_{j=0}^{n} [\phi(s_{j+1}) - \phi(s_j)] \]

\[ = \phi(s_{n+1}) - \phi(s_0) \]

\[ = \phi(b) - \phi(a) \]
**Theorem:** If $\phi$ is piecewise continuous on $[a, b]$, then

$$
\left| \int_a^b \phi(t) \, dt \right| \leq \int_a^b |\phi(t)| \, dt
$$

**Note:** By one of the special problems, $|\phi|$ is piecewise continuous on $[a, b]$ and hence Riemann integrable.
**Arc Length (Total Distance Traveled)**

**Lemma:** Let $a < b$, $\gamma : [a, b] \rightarrow \mathbb{C}$, and $\gamma$ be piecewise $C^1$. Suppose that $h_1 : [a, b] \rightarrow \mathbb{C}$ and $h_2 : [a, b] \rightarrow \mathbb{C}$ such that, for $t \in [a, b]$ with $\gamma$ differentiable at $t$,

$$h_1(t) = h_2(t) = \gamma'(t)$$

Then

- $|h_1|$ and $|h_2|$ are piecewise continuous on $[a, b]$
- $|h_1|$ and $|h_2|$ are Riemann integrable and

$$\int_a^b |h_1(t)| \, dt = \int_a^b |h_2(t)| \, dt$$

**Proof:**

- As in the just-completed proof of a complex version of the FTC, both $h_1$ and $h_2$ are piecewise continuous.
- By a special problem, $|h_1|$ and $|h_2|$ are piecewise continuous.
- By another special problem, both are Riemann integrable.
Let $S$ be the set of $t \in [a, b]$ at which $\gamma$ is not differentiable.

**Claim 1:** Suppose that $c < d$ with $[c, d] \subset [a, b]$ and $(c, d) \cap S = \emptyset$. Then

$$\int_{c}^{d} |h_1(t)| \, dt = \int_{c}^{d} |h_2(t)| \, dt$$

**Proof:** On $(c, d)$, $h_j(t) = \gamma'(t) = h_2(t)$. Hence, for $0 < \delta < (d - c)/2$, for $j = 1$ and $j = 2$ we have

$$\int_{c+\delta}^{d-\delta} |h_j(t)| \, dt = \int_{c+\delta}^{d-\delta} |\gamma'(t)| \, dt$$

Then for $j = 1$ and $j = 2$ we have

$$\int_{c}^{d} |h_j(t)| \, dt = \lim_{\delta \downarrow 0} \int_{c+\delta}^{d-\delta} |\gamma'(t)| \, dt$$
Suppose that $S \cap (a, b) = \emptyset$. Let $[c, d] = [a, b]$ in Claim 1 to conclude that

$$\int_a^b |h_1(t)| \, dt = \int_a^b |h_2(t)| \, dt$$

Now suppose that $S \cap (a, b) \neq \emptyset$.

- Enumerate $S \cap (a, b)$ as $\{s_j\}_{j=1}^n$.
- Let $a = s_0$ and $b = s_{n+1}$.
- For $0 \leq j \leq n$, $(s_j, s_{j+1}) \cap S = \emptyset$. By Claim 1,

$$\int_{s_j}^{s_{j+1}} |h_1(t)| \, dt = \int_{s_j}^{s_{j+1}} |h_2(t)| \, dt$$
Finally,

\[ \int_a^b |h_1(t)| \, dt = \sum_{j=0}^n \int_{s_j}^{s_{j+1}} |h_1(t)| \, dt = \sum_{j=0}^n \int_{s_j}^{s_{j+1}} |h_2(t)| \, dt = \int_a^b |h_2(t)| \, dt \]
The Definition of Arc Length (as Total Distance Traveled)

**Def’n:** Let $a < b$ and $\gamma : [a, b] \to \mathbb{C}$ be piecewise $C^1$. Let $h : [a, b] \to \mathbb{C}$ such that, for all $t \in [a, b]$, $\gamma$ is differentiable at $t$, then $h(t) = \gamma'(t)$. Then the length of $\gamma$ is

$$L(\gamma) = \int_a^b |h(t)| \, dt$$

In the book, Sarason writes simply $\int_a^b |\gamma'(t)| \, dt$. 