Ramsey’s Definition of Piecewise $C^1$: Let $a < b$ and $\phi : [a, b] \to \mathbb{C}$. $\phi$ is Ramsey piecewise $C^1$ if

1. $\phi$ is continuous.
2. There is a finite set $S \subset [a, b]$ such $\phi$ is differentiable at every $t \in [a, b] \setminus S$
   1. $\phi'$ is continuous at every $t \in [a, b] \setminus S$
   2. At every $t \in S \cap [a, b)$, $\phi'$ has a finite right limit.
   3. At every $t \in S \cap (a, b]$, $\phi'$ has a finite left limit.

Note: There is a subtle change that saves work. Sarason requires $\phi'$ to be continuous wherever it exists. It appears that Ramsey allows the derivative to exist but be non-continuous at a finite number of places.
Real-Valued Ramsey Piecewise $C^1$ Functions Have One-Sided Derivatives

**Lemma:** Let $a < b$ and $\phi : [a, b] \rightarrow R$ be Ramsey piecewise $C^1$. Then

1. At every $t \in [a, b)$, $\phi$ has a right derivative at $t$, labeled $\phi'(t+)$, with
   \[ \phi'(t+) = \lim_{u \downarrow t} \phi'(u) \]

2. At every $t \in (a, b]$, $\phi$ has a left derivative at $t$, labeled $\phi'(t-)$, with
   \[ \phi'(t-) = \lim_{u \uparrow t} \phi'(u) \]

**Proof:** By the definition of Ramsey piecewise $C^1$ there is a finite set $S \subset [a, b]$ such that $\phi'$ exists and is continuous on $[a, b] \setminus S$, with finite right limits of $\phi'$ at $t \in S \cap [a, b)$ and finite left limits at $t \in S \cap [a, b)$. 

Ramsey Complex Integration, Part 2
**Case 1:** Consider \( t \in [a, b] \setminus S \). \( \phi' \) exists and is continuous on \([a, b] \setminus S\) means these three things:

1. If \( a \notin S \), \( \phi \) has a right derivative at \( a \) and
   \[
   \phi'(a+) = \lim_{u \downarrow a} \phi'(u)
   \]

2. If \( b \notin S \), \( \phi \) has a left derivative at \( b \) and
   \[
   \phi'(b-) = \lim_{u \uparrow b} \phi'(u)
   \]

3. If \( t \in (a, b) \setminus S \), then \( \phi \) has a two-sided derivative at \( t \in (a, b) \) and
   \[
   \phi'(t) = \lim_{u \to t} \phi'(u)
   \]

Thus
   \[
   \phi'(t+) = \phi'(t) = \lim_{u \downarrow t} \phi'(u)
   \]

and
   \[
   \phi'(t-) = \phi'(t) = \lim_{u \uparrow t} \phi'(u)
   \]
Therefore

- By (1) and (3) above, $\phi$ has a right derivative for all $t \in [a, b) \setminus S$.
- By (2) and (3) above, $\phi$ has a left derivative for all $t \in (a, b] \setminus S$. 
**Case 2:** $t \in S$ and $t < b$. There is some $L \in \mathbb{R}$ such that

$$\lim_{u \downarrow t} \phi'(u) = L$$

Choose $\delta_0 > 0$ so that $(t, t + \delta_0) \cap S = \emptyset$ and $t + \delta_0 < b$. Given $\epsilon > 0$, there is some $\delta_1$ in $(0, \delta_0)$ so that

$$t < u < t + \delta_1 \quad \Rightarrow \quad |\phi'(u) - L| < \epsilon$$

Let $u \in (t, t + \delta_1)$. Then $\phi$ is continuous on $[t, u]$ and $\phi'$ exists on $(t, u)$. By the Mean Value Theorem, there is some $c_u \in (t, u)$ such that

$$\frac{\phi(u) - \phi(t)}{u - t} = \phi'(c_u)$$

Because $0 < |c_u - t| < |u - t| < \delta_1$, we have

$$\left| \frac{\phi(u) - \phi(t)}{u - t} - L \right| = |\phi'(c_u) - L| < \epsilon$$

Therefore, $L = \lim_{u \downarrow t} \frac{\phi(u) - \phi(t)}{u - t} = \phi'(t^+)$
Case 2: $t \in S$ and $t > a$. There is some $K \in \mathbb{R}$ such that

$$\lim_{u \uparrow t} \phi'(u) = K$$

Choose $\delta_0 > 0$ so that $(t - \delta_0, t) \cap S = \emptyset$ and $t - \delta_0 > a$. Given $\epsilon > 0$, there is some $\delta_1$ in $(0, \delta_0)$ so that

$$t - \delta_1 < u < t \implies |\phi'(u) - K| < \epsilon$$

Let $u \in (t - \delta_1, t)$. Then $\phi$ is continuous on $[u, t]$ and $\phi'$ exists on $(u, t)$. By the Mean Value Theorem, there is some $c_u \in (u, t)$ such that

$$\frac{\phi(u) - \phi(t)}{u - t} = \phi'(c_u)$$

Because $0 < |c_u - t| < |u - t| < \delta_1$, we have

$$\left| \frac{\phi(u) - \phi(t)}{u - t} - K \right| = |\phi'(c_u) - K| < \epsilon$$

Therefore, $K = \lim_{u \uparrow t} \frac{\phi(u) - \phi(t)}{u - t} = \phi'(t^-)$
**Problem VI.2.1:** Let \( a < b \) and \( \phi : [a, b] \to \mathbb{C} \) be Ramsey piecewise \( C^1 \). Then

1. At every \( t \in [a, b) \), \( \phi \) has a right derivative at \( t \), labeled \( \phi'(t+) \), with
   \[
   \phi'(t+) = \lim_{u \downarrow t} \phi'(u)
   \]

2. At every \( t \in (a, b] \), \( \phi \) has a left derivative at \( t \), labeled \( \phi'(t-) \), with
   \[
   \phi'(t-) = \lim_{u \uparrow t} \phi'(u)
   \]

**Proof:** Both \( \Re(\phi) \) and \( \Im(\phi) \) are piecewise \( C^1 \), with domains \([a, b]\) but with real ranges. Apply the lemma, to get right derivatives for both real functions on \([a, b)\) and left derivatives for both real functions on \((a, b]\).
Proof of VI.2.1 for $t \in [a, b)$

Suppose that $t \in [a, b)$. By the lemma and the definition of right derivative,

$$[\Re(\phi)]'(t+) = \lim_{u \downarrow t} \frac{\Re(\phi)(u) - \Re(\phi)(t)}{u - t} = \lim_{u \downarrow t} [\Re(\phi)]'(u)$$

and

$$[\Im(\phi)]'(t+) = \lim_{u \downarrow t} \frac{\Im(\phi)(u) - \Im(\phi)(t)}{u - t} = \lim_{u \downarrow t} [\Im(\phi)]'(u)$$

Consequently, multiplying by a constant and adding, we have

$$[\Re(\phi)]'(t+) + i \cdot [\Im(\phi)]'(t+)$$

$$= \lim_{u \downarrow t} \left[ \frac{\Re(\phi)(u) - \Re(\phi)(t)}{u - t} + i \cdot \frac{\Im(\phi)(u) - \Im(\phi)(t)}{u - t} \right]$$

$$= \lim_{u \downarrow t} \frac{\phi(u) - \phi(t)}{u - t} = \phi'(t+)$$
Continue, but next add and scale different limits:

\[ \phi'(t+) = [\Re(\phi)]'(t+) + i \cdot [\Im(\phi)]'(t+) \]
\[ = \lim_{u \downarrow t} [\Re(\phi)]'(u) + i \cdot \lim_{u \downarrow t} [\Im(\phi)]'(u) \]
\[ = \lim_{u \downarrow t} \{ [\Re(\phi)]'(u) + i \cdot [\Im(\phi)]'(u) \} \]
\[ = \lim_{u \downarrow t} \phi'(u) \]
Proof of VI.2.1 for \( t \in (a, b) \)

Suppose that \( t \in (a, b] \). Then

\[
[\Re(\phi)]'(t-) = \lim_{u \uparrow t} \frac{\Re(\phi)(u) - \Re(\phi)(t)}{u - t} \lim_{u \uparrow t} [\Re(\phi)]'(u)
\]

and

\[
[\Im(\phi)]'(t-) = \lim_{u \uparrow t} \frac{\Im(\phi)(u) - \Im(\phi)(t)}{u - t} \lim_{u \uparrow t} [\Im(\phi)]'(u)
\]

Consequently, multiplying by a constant and adding, we have

\[
[\Re(\phi)]'(t-) + i \cdot [\Im(\phi)]'(t-)
\]

\[
= \lim_{u \uparrow t} \left[ \frac{\Re(\phi)(u) - \Re(\phi)(t)}{u - t} + i \cdot \frac{\Im(\phi)(u) - \Im(\phi)(t)}{u - t} \right]
\]

\[
= \lim_{u \uparrow t} \frac{\phi(u) - \phi(t)}{u - t}
\]
Continue, but next add and scale different limits:

\[
\phi'(t+) = [\Re(\phi)]'(t+) + i \cdot [\Im(\phi)]'(t+)
\]

\[
= \lim_{u \uparrow t} [\Re(\phi)]'(u) + i \cdot \lim_{u \uparrow t} [\Im(\phi)]'(u)
\]

\[
= \lim_{u \uparrow t} \{ [\Re(\phi)]'(u) + i \cdot [\Im(\phi)]'(u) \}
\]

\[
= \lim_{u \uparrow t} \phi'(u)
\]
Theorem: Let $a < b$ and $\phi : [a, b] \rightarrow \mathbb{C}$. Then $\phi$ is Sarason piecewise $C^1$ if and only if it is Ramsey piecewise $C^1$.

Proof: The direction $\Rightarrow$ is immediate, because the Sarason definition requires more continuity. The proof of the converse consists of establishing continuity of $\phi'$ at those places in $S$ where $\phi'$ does exist.

Suppose that $a < b$, and $\phi : [a, b] \rightarrow \mathbb{C}$ is Ramsey piecewise $C^1$. Then there is a finite set $S \subset [a, b]$ such that $\phi'$ exists and is continuous on $[a, b] \setminus S$ with finite right limits of $\phi'$ at $t \in S \cap [a, b)$ and finite left limits at $t \in S \cap (a, b]$. 

Suppose that $a \in S$. Then $\phi'(a) = \phi'(a^+)$ by definition of differentiable on a closed interval. By the Ramsey version of Problem VI.2.1,

$$\phi'(a^+) = \lim_{u \downarrow a} \phi'(u)$$

This makes $\phi'$ continuous at $a$, by the definition of continuity on a closed interval.

Note that the preceding argument implies that we are never forced to have $a \in S$. The definitions automatically gives us a derivative at $a$ and continuity of the derivative at $a$. 
Suppose that \( b \in S \). Then \( \phi'(b) = \phi'(b^-) \) by definition of differentiable on a closed interval. By the Ramsey version of Problem VI.2.1,

\[
\phi'(b^-) = \lim_{u \uparrow a} \phi'(u)
\]

This makes \( \phi' \) continuous at \( b \), by the definition of continuity on a closed interval.

Note that the preceding argument implies that we are never forced to have \( b \in S \). The definitions automatically gives us a derivative at \( b \) and continuity of the derivative at \( b \).
Suppose that $t \in (a, b) \cap S$.

- By the Ramsey version of VI.2.1, both $\phi'(t+)$ and $\phi'(t-)$ exist and

  $$
  \phi'(t+) = \lim_{u \downarrow t} \phi'(u) \quad \text{and} \quad \phi'(t-) = \lim_{u \uparrow t} \phi'(u)
  $$

- $\phi'(t)$ exists if and only if $\phi'(t+) = \phi'(t-)$. When $\phi'(t)$ exists,

  $$
  \phi'(t) = \phi'(t+) = \phi'(t-)
  $$

- Consequently, if $\phi'(t)$ exists,

  $$
  \phi'(t) = \lim_{u \downarrow t} \phi'(u) \quad \text{and} \quad \phi'(t) = \lim_{u \uparrow t} \phi'(u)
  $$

Thus, $\phi'$ is continuous at $t$ in the two-sided sense.
**Theorem:** Let \( a < b \) and \( \phi : [a, b] \to \mathbb{C} \). \( \phi \) is piecewise \( C^1 \) if and only if, there is a finite set \( S \subset (a, b) \), which we enumerate as \( s_1 < s_2 < \ldots < s_n \), such that \( \phi \) restricted to each \([s_j, s_{j+1}]\) is \( C^1 \) in the following sense:

- \( \phi \) has a two-sided derivative at each \( t \in (s_j, s_{j+1}) \), which we label \( \phi'(t) \)
- \( \phi \) has a right derivative at \( s_j \), which we label \( \phi'(s_j+) \)
- \( \phi \) has a left derivative at \( s_{j+1} \), which we label \( \phi'(s_{j+1}-) \)
- \( \phi' \) is continuous on \((s_j, s_{j+1})\)
- The right limit of \( \phi'(t) \) at \( s_j \) equals \( \phi'(s_j+) \)
- The left limit of \( \phi'(t) \) at \( s_{j+1} \) equals \( \phi'(s_{j+1}-) \)