BOHR CLUSTER POINTS OF SIDON SETS

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July 19, 1994

Abstract. If there is a Sidon subset of the integers Z which has a member of Z as a cluster point in the Bohr compactification of Z, then there is a Sidon subset of Z which is dense in the Bohr compactification. A weaker result holds for quasi-independent and dissociate subsets of Z.

It is a long standing open problem whether Sidon subsets of Z can be dense in the Bohr compactification of Z ([LR]). Yitzhak Katznelson came closest to resolving the issue with a random process in which almost all sets were Sidon and and almost all sets failed to be dense in the Bohr compactification [K]. This note, which does not resolve this open problem, supplies additional evidence that the problem is delicate: it is proved here that if one has a Sidon set which clusters at even one member of Z, one can construct from it another Sidon set which is dense in the Bohr compactification of Z. A weaker result holds for quasi-independent and dissociate subsets of Z.

Cluster Points. By the definition of the Bohr topology, a subset $E \subset Z$ clusters at q if and only if, for all $\epsilon \in R^+$, for all $n \in Z^+$, and for all $(t_1, \dots, t_n) \in T^n$, there is some $m \in E$ such that

$$\sup_{1 \le i \le n} |< m, t_i > - < q, t_i > | < \epsilon. \tag{1}$$

Here T is the dual group of Z and $< m, t >$ denotes the result of the character m acting on t. Thus, if T is represented as $[-\pi, \pi)$ with addition mod 2π,

$$< m, t >= e^{imt}.$$

If, for all $(t_1, \dots, t_n) \in T^n$, there is at least one $m \in E$ such that inequality (1) holds, then E is said to **approximate q within ϵ on T^n**.

1991 *Mathematics Subject Classification.* 43A56.

Key words and phrases. Sidon, Bohr compactification, quasi-independent, dissociate.

I THANK KEN ROSS AND KATHRYN HARE FOR THEIR HELPFUL CORRECTIONS OF AN EARLY VERSION OF THIS MANUSCRIPT.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Overview. Let E be a Sidon subset of the integers Z which clusters at the integer $q \in Z$ in the topology of the Bohr compactification. The dense Sidon set will have the form

$$S = \cup_{j=1}^{\infty} S_j, \quad \text{with } S_j = x_j + k_j(E_j - q),$$

where $E_j \subset E$ approximates q within $1/m_j$ on T^{n_j} under an exhaustive enumeration (x_j, n_j, m_j) of $Z \times Z^+ \times Z^+$. Lemma 1 below asserts that finite $E_j \subset E$ can always be found. Lemma 3 below says that S is dense, regardless of the dilation factors k_j. The final step of the argument is to choose k_j's so that S is Sidon. Lemma 4 does this in part for N-independent sets (N-independent generalizes quasi-independent and dissociate; it is defined below). It is then a short step to Sidon sets, using a criterion of Gilles Pisier's.

Lemma 1 (Compactness). *Let $E \subset Z$ cluster at $q \in Z$ in the topology of the Bohr compactification of Z. For every $\epsilon \in R^+$ and $n \in Z^+$, there is a finite subset $E' \subset E$ which approximates q within ϵ on T^n.*

Proof. Let $\epsilon \in R^+$ and $n \in Z^+$ be given. For each $(t_1, \ldots, t_n) \in T^n$ there is some $m \in E$ such that (1) holds with $\epsilon/2$ in the role of ϵ. By the continuity of the characters m and q on T (both are in Z), there is an open neighborhood U of $(t_1, \ldots, t_n) \in T^n$ for which (1) is valid when $(s_1, \ldots, s_n) \in U$ are substituted for $(t_1, \ldots, t_n)$. By the compactness of T^n, a finite number of such U's cover T^n. The set of m's corresponding to the U's can be taken for the set E'. $\square$

For integers k, y, and z, and for $S \subset Z$, let $z + k(S - y)$ denote $\{z + k(x - y) \mid x \in S\}$.

Lemma 2 (Dilation). *Let k, y, and z be integers. If S approximates y within ϵ on T^n, then $z + k(S - y)$ approximates z within ϵ on T^n.*

Proof. Let $(t_1, \ldots, t_n) \in T^n$. There is some $m \in S$ such that

$$\sup_{1 \le i \le n} | < m, kt_i > - < y, kt_i > | < \epsilon.$$

Because m and k are integers, $< m, kt > = < mk, t >$. Therefore,

$$\begin{aligned} | < z + k(m - y), t_i > - < z, t_i > | &= | < z - ky, t_i > (< km, t_i > - < ky, t_i >)| \\ &= | < m, kt_i > - < y, kt_i > | \\ &< \epsilon, \end{aligned}$$

for $1 \le i \le n$. $\square$

Lemma 3 (Denseness). *Let (x_j, n_j, m_j), $j \in Z^+$, exhaustively enumerate*

$$\{(x, n, m) \mid x \in Z, n \in Z^+, m \in Z^+\}.$$

Suppose there is a sequence $\{E_j\}_{j=1}^{\infty}$ of subsets of Z such that E_j approximates p_j within $1/m_j$ on T^{n_j}. Then for any sequence of integers k_j, $S = \cup_{j=1}^{\infty}(x_j + k_j(E_j - p_j))$ is dense in the Bohr compactification of Z.

Proof. Since Z is dense in its Bohr compactification, it suffices to show that the closure of S includes every $x \in Z$. Let $x \in Z$. By the definition of the Bohr topology, we must show that S approximates x within ϵ on T^n for any $\epsilon \in R^+$ and any $n \in Z^+$. Choose some $m \in Z^+$ such that $1/m < \epsilon$. The triple (x, n, m) is (x_j, n_j, m_j) for some j. Since E_j approximates p_j with $1/m_j$ on T^{n_j}, the Dilation Lemma implies that $x_j + k_j(E_j - p_j)$ approximates x_j within $1/m_j$ on T^{n_j} and hence x within ϵ on T^n. $\square$

Definitions. *Let N be a positive integer and G be an additive group. An N-relation is a linear combination*

$$\sum_{x\in G} \alpha_x x = 0,$$

where α_x an integer in $[-N, N]$ for all x and with $\alpha_x \neq 0$ for at most finitely many x. A subset A of G is said to be N-independent if and only if the only N-relation among its elements is the trivial relation which has all coefficients equal to 0. The N-relation hull of A, $[A]_N$, is

$$\{\sum_{x\in A} \alpha_x x \mid \alpha_x \in \{-N, -N+1, \dots, N\}\}.$$

The hull of the empty set is understood to be $\{0\}$.[1]

Quasi-independent sets are the 1-independent sets, while dissociate sets are the 2-independent sets ([P], [LR]).

Lemma 4. *Let $\{W_j\}_{j=1}^{\infty}$ be a sequence of finite N-independent subsets of Z. Let x_j be arbitrary integers, $1 \le j < \infty$. Set D_j equal to the maximum absolute value of the elements of*

$$[\cup_{i<j}(x_i + k_i W_i)]_N,$$

and let M_j denote the size of W_j. If $k_j > D_j + NM_j|x_j|$ for all $j \ge 1$, then

$$\cup_{j=1}^{\infty}(x_j + k_j W_j)$$

is N-independent. Moreover, the sets $x_j + k_j W_j$ are disjoint for distinct values of j.

Proof. Let $W_i' = x_i + k_i W_i$, and set

$$V_j = \cup_{i<j} W_i'.$$

Since $V_1 = \emptyset$, it is certainly N-independent. Assume that V_j is N-independent for some $j \ge 1$, and that W_{i_1}' and W_{i_2}' are disjoint for $i_1 \neq i_2$ with $i_1 < j$ and $i_2 < j$. Consider V_{j+1}. It will be proved first that W_j' is disjoint from V_j. Let $x \in W_j'$ and $y \in V_j$. Then $x = x_j + k_j x'$ for some $x' \in W_j$. Since W_j is N-independent, $0 \notin W_j$ and thus $x' \neq 0$. Therefore, since $V_j \subset [V_j]_N$,

$$|x| = |x_j + k_j x'| \ge k_j - |x_j| > D_j + NM_j|x_j| - |x_j| \ge D_j \ge |y|.$$

Next, consider an N-relation on V_{j+1} with coefficients α_x for $x \in V_{j+1}$. Since W_j' is disjoint from V_j, one may write

$$\sum_{x\in W_j'} \alpha_x x = -\sum_{x\in V_j} \alpha_x x = \tau,$$

[1] This definition is distinct from that of J. Bourgain, who defined N-independence to be weaker versions of quasi-independence.

for some $\tau \in [V_j]_N$. Each $x \in W_j'$ has the form $x_j + k_j x'$ for some x' in W_j (x' is unique since $k_j > 0$). Thus,

$$k_j \sum_{x \in W_j'} \alpha_x x' = \tau - x_j \sum_{x \in W_j'} \alpha_x. \tag{2}$$

Suppose that $\sum_{x \in W_j'} \alpha_x x' \neq 0$. Then, by equation (2),

$$\begin{aligned} k_j &\leq \left| k_j \sum_{x \in W_j'} \alpha_x x' \right| \\ &= \left| \tau - x_j \sum_{x \in W_j'} \alpha_x \right| \\ &\leq |\tau| + |x_j| \cdot \left| \sum_{x \in W_j'} \alpha_x \right| \\ &\leq D_j + |x_j| N M_j, \end{aligned}$$

which is contrary to $k_j > D_j + NM_j|x_j|$. Thus $\sum_{x \in W_j'} \alpha_x x' = 0$. This is an N-relation among the elements of W_j (since x' is unique for each x, and vice versa). Since W_j is N-independent, $\alpha_x = 0$ for $x \in W_j'$. It follows that equation (2) reduces to $\tau = 0$, which is an N-relation supported on V_j. Since V_j is N-independent, $\alpha_x = 0$ for for all $x \in V_j$ and hence for all $x \in V_{j+1} = V_j \cup W_j'$. Thus only the trivial relation occurs among the N-relations on V_{j+1}.

Finally, since $V_j \subset V_{j+1}$ for all $j \in Z^+$ and

$$S = \cup_{i=1}^{\infty} W_i' = \cup_{j=1}^{\infty} V_j,$$

the N-independence of the V_j's makes S be N-independent. [Any N-relation on S has at most finitely many non-zero coefficients (by definition); thus it must be supported on V_j for some j (since S is an increasing union of the V_j's) and hence is trivial because V_j is N-independent.] □

Proposition 5. *If there is a Sidon set E which clusters at some $n \in Z$ in the topology of the Bohr compactification of Z, then there is a Sidon set which is dense in the Bohr compactification of Z.*

Proof. By Lemma 2, $E' = E - n$ clusters at 0 in the Bohr topology; it is well known that E' is Sidon, in fact with the same Sidon constant as E ([LR]). By the definition of cluster point, we may assume $0 \notin E'$. As provided by Lemma 1, for any positive integers n and m there are finite subsets $E(n,m) \subset E'$ such that $E(n,m)$ approximates 0 within $1/m$ on T^n. As in Lemma 3, with $p_j = 0$, $E_j = E(n_j, m_j)$, and k_j yet to be determined, let

$$S = \cup_{j=1}^{\infty} (x_j + k_j E_j).$$

Then S is dense in the Bohr compactification of Z.

It remains to be seen that S is Sidon, provided the k_j's are chosen well. Let the k_j's satisfy this criterion: $k_j > D_j + M_j|x_j|$ (as in Lemma 4), where M_j is the size of E_j (which is the same size as $x_j + k_jE_j$) and D_j is the maximum absolute value of the elements of

$$[\cup_{i<j}(x_i + k_iE_i)]_N.$$

This by itself guarantees that the sets $x_j + k_jE_j$ are disjoint for distinct values of j. To see this, consider $w \in x_j + k_jE_j$ and $\tau \in x_i + k_iE_i$ for $i < j$. Then $|\tau| \leq D_j$ while, because $0 \notin E'$ and hence $0 \notin E_j \subset E'$, there is some $x \neq 0$ such that

$$|w| = |x_j + k_jx| \geq k_j - |x_j| > D_j \geq |\tau|.$$

Gilles Pisier discovered the following arithmetic condition for Sidonicity ([P]). Let $|H|$ denote the cardinality of H. A set Q is Sidon if and only if there is some $\lambda \in (0,1)$ such that, for every finite subset H of Q, there is a subset F of H such that F is a quasi-independent and $|F| \geq \lambda|H|$. Let λ satisfy this criterion for the set E'.

It will be shown that λ also works for S. Let H be any finite subset of S. Then $H_j = H \cap (x_j + k_jE_j)$ is finite for each j; by the second paragraph of this proof, the H_j's are disjoint and thus

$$|H| = \sum_{j=1}^{\infty} |H_j|.$$

Since $k_j > 0$, $H_j = x_j + k_jH'_j$ and $|H'_j| = |H_j|$ for some $H'_j \subset E_j$. Recall that $E_j = E(n_j, m_j) \subset E'$. There is some $F'_j \subset H'_j$ such that F'_j is quasi-independent and $|F'_j| \geq \lambda|H'_j|$. Let

$$F = \cup_{j=1}^{\infty}(x_j + k_jF'_j).$$

Note that $M_j = |E_j| \geq |F'_j|$ and that D_j dominates the largest absolute value of

$$[\cup_{i<j}(x_i + k_iF'_i)]_N \subset [\cup_{i<j}(x_i + k_iE_i)]_N.$$

Thus the k_j's grow fast enough to allow Lemma 4 to apply to F with $N = 1$: F is quasi-independent and the sets $x_j + k_jF'_j$ are disjoint. Thus, $F \subset H$ and

$$\begin{aligned}
|F| &= \sum_{j=1}^{\infty} |x_j + k_jF'_j| \\
&= \sum_{j=1}^{\infty} |F'_j|, \quad \text{since } k_j > 0, \\
&\geq \lambda \sum_{j=1}^{\infty} |H'_j| \\
&= \lambda \sum_{j=1}^{\infty} |H_j| \\
&= \lambda|H|.
\end{aligned}$$

It follows that S is at least as Sidon as E' according to Gilles Pisier's criterion. □

The proof given above is easily modified for the N-independent sets. One of the early steps in the proof for Sidon sets does not work: when E is N-independent, $E - n$ need not be N-independent. For that reason, the theorem is weaker.

Proposition 6. *Let $E \subset Z$ be an N-independent set which clusters at* 0 *in the Bohr compactification of Z. Then there is an N-independent subset $E' \subset Z$ which is dense in the Bohr compactification of Z.*

Proof. The N-independence of E excludes 0 from E. From this point, the proof for Sidon sets is easily adapted. One chooses $k_j > D_j + M_j N|x_j|$. Then S is dense in the Bohr group as before and the rest of the proof becomes easier. There is no need to consider a finite subset $H \subset S$. The choice of $k_j > D_j + M_j N|x_j|$ and Lemma 4 directly imply that S is N-independent. $\square$

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