



**Solution of Harder Problem.** From right to left, let the columns be 1, 2, 3, 4, 5. The sum of the rightmost column 1 (above the double line) is at most  $9 + 9 + 9 + 8 + 8 = 43$  and thus the carry over is at most 4. Let the carry over above column  $i$  be  $o_i$ . We have  $o_1 = 0$ , and we have just seen that  $o_2 \leq 4$ . We have the relations

$$\begin{aligned} 3E + 2O &= N + 10 \cdot o_2, \\ 2E + 2W + N + o_2 &= E + 10 \cdot o_3, \\ 2R + 2T + O + o_3 &= V + 10 \cdot o_4, \\ 2H + o_4 &= E + 10 \cdot o_5 \quad \text{and} \\ 2T + o_5 &= L + 10 \cdot E. \end{aligned}$$

Thus,

$$E + 10 \cdot o_3 \leq 2 \cdot 9 + 9 + 2 \cdot 8 + 7 + 4 = 45 \Rightarrow o_3 \leq 4$$

$$V + 10 \cdot o_4 \leq 2 \cdot 9 + 2 \cdot 8 + 7 + 4 = 45 \Rightarrow o_4 \leq 4$$

$$E + 10 \cdot o_5 \leq 2 \cdot 9 + 4 = 22 \Rightarrow o_5 \leq 2, \text{ and } E = 1 \text{ or } E = 2.$$

$$L + 10 \cdot E = 2T + o_5 \leq 2 \cdot 9 + 2 = 20 \Rightarrow E = 1, \text{ or } (E = 2, T = 9, o_5 = 2 \text{ and } L = 0).$$

**Claim 1.**  $E = 1$ , and  $o_4 = 1$  or  $3$ .

**Proof.** Assuming otherwise, that  $E = 2$ , we have just seen that then  $T = 9$  and  $o_5 = 2$ . Since  $T = 9$ , we have  $O \leq 8$ . Then

$$N + 10 \cdot o_2 = 3E + 2 \cdot O = 6 + 2 \cdot O \leq 22,$$

so that  $o_2 \leq 2$ . Since  $T = 9$ , we have  $W \leq 8$  and  $2W + N \leq 16 + 7 = 23$ , and

$$E + 10 \cdot o_3 = o_2 + 2E + 2W + N \leq 2 + 2 \cdot 2 + 23 = 29,$$

and so  $o_3 \leq 2$ . Then

$$V + 10 \cdot o_4 = 2R + 2T + O + o_3 \leq 2 \cdot 8 + 2 \cdot 9 + 2 = 36,$$

and so  $o_4 \leq 3$ . Thus,  $E + 10 \cdot o_5 = 2H + o_4 \leq 2 \cdot 8 + 3 = 19$ , and so  $o_5 \leq 1$  which contradicts  $o_5 = 2$  already found above. Finally,

$$2H + o_4 = E + 10 \cdot o_5 \Rightarrow o_4 \text{ is odd,}$$

in which case  $o_4 = 1$  or  $3$ , since we know  $o_4 \leq 4$ . ■

**Claim 2.**  $o_2 = 0$ , and  $O = 2$  or  $O = 3$ .

**Proof.** Since  $E = 1$ , we have

$$3 + 2O = 3E + 2O = N + 10 \cdot o_2 \Rightarrow N \text{ is odd.}$$

We have

$$2 + 2W + N + o_2 = 2E + 2W + N + o_2 = E + 10 \cdot o_3 = 1 + 10 \cdot o_3,$$

and hence  $o_2$  is even (since  $N$  is odd). Since  $N + 10 \cdot o_2 = 3 + 2O \leq 21$ , we have  $o_2 = 0$ , as  $o_2 = 2$  yields  $N + 10 \cdot o_2 = 21$ , whence  $N = 1 = E$  (impossible). Thus,

$$N = N + 10 \cdot o_2 = 3E + 2O = 3 + 2O \Rightarrow (O = 2 \text{ and } N = 7) \text{ or } (O = 3 \text{ and } N = 9).$$

since  $O \neq E = 1$ , and  $O \neq 0$  since the replacement for ONE would begin with 0. ■

**Claim 3.**  $O = 3$  and  $N = 9$ .

**Proof.** Assume otherwise that  $O = 2$  and hence  $N = 7$ . Then

$$\begin{aligned} 2E + 2W + N + o_2 &= E + 10 \cdot o_3 \Rightarrow 2 + 2W + 7 = 1 + 10 \cdot o_3 \\ &\Rightarrow 2W + 8 = 10 \cdot o_3 \Rightarrow o_3 = 1 \text{ or } 2, \end{aligned}$$

since  $0 \leq 2W + 8 \leq 27$ . Now,

$$o_3 = 1 \Rightarrow 2W + 8 = 10 \Rightarrow W = 1 = E \text{ (impossible),}$$

and

$$o_3 = 2 \Rightarrow 2W + 8 = 20 \Rightarrow W = 6.$$

Now,

$$2R + 2T + O + o_3 = V + 10 \cdot o_4 \Rightarrow 2(R + T) + 4 = V + 10 \cdot o_4.$$

Note that

$$\begin{aligned} o_4 = 1 &\Rightarrow 2H + o_4 = 2H + 1 = 10o_5 + 1 \Rightarrow H = 5o_5 \\ &\Rightarrow (H = 0 \text{ and } o_5 = 0) \text{ or } (H = 5 \text{ and } o_5 = 1). \end{aligned}$$

We show that “ $o_4 = 1, H = 0$  and  $o_5 = 0$ ” cannot hold.

$$\begin{aligned} (o_4 = 1, H = 0 \text{ and } o_5 = 0) &\Rightarrow 2T = 2T + o_5 = L + 10 \cdot E = L + 10 \\ &\Rightarrow 2T = 10 + L \Rightarrow L = (0, 2, 4, 6 \text{ or } 8) \text{ and } T = (5, 6, 7, 8, \text{ or } 9), \text{ respectively.} \end{aligned}$$

Note that  $L \neq 0$ , since  $H = 0$ . Also,  $L \neq 2$ , since  $O = 2$ . Moreover,  $L \neq 4$ , since then  $T = 7$ , but  $O = 2 \Rightarrow N = 7$ . Now,  $L \neq 6$  since  $W = 6$ . Thus  $L = 8$  and  $T = 9$ . From above,

$$\begin{aligned} 2(R + T) + 4 = V + 10 \cdot o_4 &\Rightarrow 2(R + 9) + 4 = V + 10 \\ &\Rightarrow 2R + 22 = V + 10 \Rightarrow V = 2R + 12 \text{ (impossible)} \end{aligned}$$

Hence “ $o_4 = 1, H = 0$  and  $o_5 = 0$ ” is impossible. We show that “ $o_4 = 1, H = 5$  and  $o_5 = 1$ ” is also impossible:

$$\begin{aligned} (o_4 = 1, H = 5 \text{ and } o_5 = 1) &\Rightarrow 2T + 1 = 2T + o_5 = L + 10 \cdot E = L + 10 \\ &\Rightarrow 2T = L + 9 \Rightarrow L = (1, 3, 5, 7, \text{ or } 9), \text{ and } T = (5, 6, 7, 8, \text{ or } 9), \text{ respectively.} \end{aligned}$$

Now  $L \neq 1$  since  $E = 1$ . Also  $L \neq 3$ , since  $L = 3 \Rightarrow T = 6 = W$ . Moreover,  $L \neq 5$ , since  $H = 5$ . Since  $N = 7, L \neq 7$ . Finally  $L \neq 9$ , since  $L = 9 \Rightarrow T = 9$  (impossible). Hence  $O = 2$  leads to a contradiction, and so  $O = 3$  and  $N = 9$ .  
■

**Claim 4.**  $W = 0$  and  $o_3 = 1$ .

**Proof.** We have,

$$\begin{aligned} 2E + 2W + N + o_2 = E + 10 \cdot o_3 &\Rightarrow 2 + 2W + 9 = 1 + 10 \cdot o_3 \\ &\Rightarrow 2W + 10 = 10 \cdot o_3 \Rightarrow W = 5 \cdot (o_3 - 1) \\ &\Rightarrow (W = 0 \text{ and } o_3 = 1) \text{ or } (W = 5 \text{ and } o_3 = 2). \end{aligned}$$

Assuming that  $W = 5$  and  $o_3 = 2$ , we will produce a contradiction. Note that

$$\begin{aligned} 2R + 2T + O + o_3 = V + 10 \cdot o_4 &\Rightarrow 2R + 2T + 3 + 2 = V + 10 \cdot o_4 \\ &\Rightarrow 2(R + T) + 5 = V + 10 \cdot o_4. \end{aligned}$$

We know that  $o_4 = 1$  or  $3$ .

Assume first that  $o_4 = 1$ . Then

$$\begin{aligned} 2H + o_4 = 1 + 10 \cdot o_5 &\Rightarrow 2H = 10 \cdot o_5 \Rightarrow H = 5 \cdot o_5 \\ &\Rightarrow H = 0 \text{ and } o_5 = 0 \text{ (since } H \neq W = 5\text{)}. \end{aligned}$$

Moreover,

$$2T + o_5 = L + 10 \cdot E = L + 10 \Rightarrow 2T = L + 10 \Rightarrow T \geq 6,$$

since  $T \neq W = 5$ . Now, from  $T \geq 6$  and  $R \geq 2$  (since  $H = 0$  and  $E = 1$ )

$$\begin{aligned} 2(R + T) + 5 = V + 10 \cdot o_4 = V + 10 \\ \Rightarrow V = 2(R + T) - 5 \geq 2R + 7 \geq 4 + 7 = 11 \text{ (impossible)}. \end{aligned}$$

Thus,  $o_4 = 1$  is not possible. Thus, “ $W = 5$  and  $o_3 = 2$ ” is not possible.

Assume now that  $o_4 = 3$ . Then

$$2(R + T) + 5 = V + 10 \cdot o_4 = V + 30.$$

Moreover,

$$\begin{aligned} 2H + 3 = 2H + o_4 = 1 + 10 \cdot o_5 \\ \Rightarrow 2H + 2 = 10 \cdot o_5 = 10 \text{ or } 20 \Rightarrow H = 4 \text{ or } H = 9 = N \Rightarrow H = 4. \end{aligned}$$

Thus,  $H = 4$  and  $o_5 = 1$ .

$$2T + o_5 = L + 10 \cdot E \Rightarrow 2T + 1 = L + 10 \Rightarrow L \text{ odd} \Rightarrow L = 7,$$

since  $E = 1, O = 3, W = 5$  and  $N = 9$ . Then  $2T + 1 = L + 10 = 17 \Rightarrow T = 8$ .

$$\begin{aligned} 2(R + T) + 5 = V + 10 \cdot o_4 = V + 30 \Rightarrow 2(R + 8) + 5 = V + 10 \cdot o_4 = V + 30 \\ \Rightarrow 2R + 21 = V + 30 \Rightarrow 2R = V + 9 \Rightarrow V \text{ odd}. \end{aligned}$$

However, all of the odds are taken:  $E = 1, O = 3, W = 5, L = 7$  and  $N = 9$ . Thus,  $o_4 = 3$  is also not possible. ■

**Claim 5.**  $o_4 = 3$ .

**Proof.** Assume otherwise that  $o_4 = 1$ .

$$\begin{aligned} 2H + o_4 = 1 + 10 \cdot o_5 &\Rightarrow 2H + 1 = 1 + 10 \cdot o_5 \Rightarrow H = 5 \cdot o_5 \\ &\Rightarrow (H = o_5 = 0) \text{ or } (H = 5 \text{ and } o_5 = 1) \\ &\Rightarrow H = 5 \text{ and } o_5 = 1, \end{aligned}$$

since  $W = 0$ . Moreover, since  $o_3 = 1$ ,

$$\begin{aligned} 2R + 2T + O + o_3 &= V + 10 \cdot o_4 \Rightarrow 2R + 2T + 4 = V + 10 \\ 2R + 2T &= V + 6. \end{aligned}$$

Now,

$$2T + o_5 = L + 10 \Rightarrow 2T + 1 = L + 10 \Rightarrow 2T = L + 9 \geq 11 \Rightarrow T \geq 6,$$

and so

$$V + 6 = 2R + 2T \geq 2R + 12, \text{ or } V \geq 2R + 6 \geq 10 \text{ (impossible).}$$

Thus,  $o_4 = 3$ . ■

**Claim 6.**  $H = 4$  and  $o_5 = 1$ .

**Proof.**

$$2H + o_4 = 1 + 10 \cdot o_5 \Rightarrow 2H + 3 = 1 + 10 \cdot o_5 \Rightarrow H + 1 = 5 \cdot o_5$$

We know  $o_5 = 0, 1$ , or  $2$ .

$$o_5 = 0 \Rightarrow H + 1 = 0 \text{ (impossible)}$$

$$o_5 = 1 \Rightarrow H + 1 = 5 \Rightarrow H = 4$$

$$o_5 = 2 \Rightarrow H + 1 = 10 \Rightarrow H = 9 \text{ (impossible)}$$

Thus,  $H = 4$  and  $o_5 = 1$ . ■

**Claim 7.**  $T \geq 6$  and  $V = 2$ .

**Proof.** Note that

$$\begin{aligned} 2T + o_5 &= L + 10 \Rightarrow 2T + 1 = L + 10 \\ &\Rightarrow 2T = L + 9 \geq 11 \Rightarrow T \geq 6. \end{aligned}$$

Since  $o_3 = 1$  and  $o_4 = 3$ ,

$$\begin{aligned} 2R + 2T + O + o_3 &= V + 10 \cdot o_4 \\ &\Rightarrow 2R + 2T + O + 1 = V + 10 \cdot 3 \\ &\Rightarrow 2R + 2T + 4 = V + 30 \Rightarrow 2R + 2T = V + 26. \end{aligned}$$

