

Problems of the Month for UH Mānoa Undergraduates
Solutions for November 2007

Problem A. Note that $\sqrt{2}$ is not the quotient $\frac{m}{n}$ of two integers m and n . Indeed, $\sqrt{2} = \frac{m}{n}$ implies that $2n^2 = m^2$ which implies that m has a factor of 2, say $m = 2m'$ for some integer m' . Then $2n^2 = m^2 = 4(m')^2$ and $n^2 = 2(m')^2$, in which case n has a factor of 2, say $n = 2n'$. Then $\frac{m}{n} = \frac{m'}{n'}$, and continuing the argument, m and n have infinitely many factors of 2, which is impossible. Show that there are two real numbers x and y (possibly equal) neither of which is a quotient of two integers, such that x^y is a quotient of two integers. **Hint.** You need not find values for x and y . Just provide a short proof that such numbers exist. There is a proof which is only a few lines long!

Problem B. Recall that $2! = 2$, $3! = 3 \cdot 2 = 6$, and for any integer $n > 3$, $n! = n \cdot (n - 1)!$. For $n \geq 4$, let $P_n(x)$ be the polynomial defined by

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

Show that for all $n > 4$, there is no polynomial $Q_n(x)$ and real number a_n , such that

$$P_n(x) = (x - a_n)^2 Q_n(x).$$

Solution of Problem A. First try $x = y = \sqrt{2}$. If $(\sqrt{2})^{\sqrt{2}}$ is the quotient of integers (i.e., rational), then we are done. If $(\sqrt{2})^{\sqrt{2}}$ is not rational, then let $x = (\sqrt{2})^{\sqrt{2}}$ and $y = \sqrt{2}$. We were not done before, we are now, since

$$x^y = \left((\sqrt{2})^{\sqrt{2}} \right)^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2} \cdot \sqrt{2}} = (\sqrt{2})^2 = 2 = \frac{2}{1}.$$

Solution of Problem B. Assume that $P_n(x) = (x - a_n)^2 Q_n(x)$. Then

$$P'_n(x) = 2(x - a_n)Q_n(x) + (x - a_n)^2 Q'_n(x).$$

Thus, $P'_n(a_n) = 0$, and of course $P_n(a_n) = 0$. Note also that

$$\begin{aligned} P'_n(x) &= \frac{d}{dx} \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n \right) = 0 + 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{n}{n!}x^{n-1} \\ &= 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{(n-1)!}x^{n-1} = P_{n-1}(x). \end{aligned}$$

Observe also that $P_n(x) = P_{n-1}(x) + \frac{1}{n!}x^n$. Then

$$0 = P_n(a_n) = P_{n-1}(a_n) + \frac{1}{n!}(a_n)^n = P'_n(a_n) + \frac{1}{n!}(a_n)^n = 0 + \frac{1}{n!}(a_n)^n = \frac{1}{n!}(a_n)^n.$$

Thus, $a_n = 0$. However, then $0 = P_n(a_n) = P_n(0) = 1$, a contradiction. Thus, $P_n(x) = (x - a_n)^2 Q_n(x)$ is false.