Problems of the Month for UH Mānoa Undergraduates
Solutions for September 2007

Problem A. A rational number is a real number which can be expressed as a fraction m/n where both m and n are integers (i.e., 0, ± 1, ± 2, $\pm 3, \ldots$) with $n \neq 0$. Assume that x and y are both rational numbers, $x^2 + y^2 = 1$, and $(x, y) \neq (0, 1)$. Prove that there is exactly one rational number r, such that

$$(x, y) = \left(\frac{2r}{r^2 + 1}, \frac{r^2 - 1}{r^2 + 1} \right).$$

Problem B. In an ordinary game of tennis (not a "tie breaker" game), the first player to win 4 points wins the game, provided s/he is ahead by at least two points. Thus, if each player has won 3 points, play continues until one of the players is two points ahead. Suppose that Player #1 wins any given *point* against Player #2 with a fixed probability p $(0 \leq p \leq 1)$. What is the probability $f(p)$ (as a function of p) that Player #1 wins a given *game* against Player #2? Show the derivation of your answer instead of just stating it. As a test of your answer for $f(p)$, it can simplified to the form polynomial of degree 7 divided by a polynomial of degree 2.

Solution of Problem A. The unit circle is

$$S^1 := \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \right\}.$$

Note that for any $r \in \mathbb{R}$,

$$\left(\frac{2r}{r^2 + 1}, \frac{r^2 - 1}{r^2 + 1} \right) \in S^1$$

since

$$\left(\frac{2r}{r^2 + 1} \right)^2 + \left(\frac{r^2 - 1}{r^2 + 1} \right)^2 = \frac{4r^2 + (r^4 - 2r^2 + 1)}{(r^2 + 1)^2} = \frac{r^4 + 2r^2 + 1}{(r^2 + 1)^2} = 1.$$

Let $\widetilde{S}^1 = S^1$ with the point $(0, 1)$ removed. Since $\frac{r^2 - 1}{r^2 + 1} \neq 1$ for all $r \in \mathbb{R}$ we may define

$$F : \mathbb{R} \to \widetilde{S}^1 \text{ by } F(r) := \left(\frac{2r}{r^2 + 1}, \frac{r^2 - 1}{r^2 + 1} \right).$$

We attempt to find an inverse for F. Suppose that $(x, y) \in \widetilde{S}^1$ and

$$(x, y) = F(r) = \left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1}\right).$$

Then

$$\begin{aligned} 1 - y = 1 - \frac{r^2-1}{r^2+1} &= \frac{r^2+1}{r^2+1} - \frac{r^2-1}{r^2+1} = \frac{2}{r^2+1} \\ \Rightarrow (1-y)r &= \frac{2r}{r^2+1} = x \Rightarrow r = \frac{x}{1-y}, \end{aligned}$$

Thus, let

$$G : \widetilde{S}^1 \to \mathbb{R} \text{ be defined by } G(x, y) = \frac{x}{1-y}.$$

Note that

$$G(F(r)) = G\left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1}\right) = \frac{\frac{2r}{r^2+1}}{1 - \frac{r^2-1}{r^2+1}} = \frac{2r}{r^2+1-(r^2-1)} = r.$$

Moreover,

$$F(G(x, y)) = F\left(\frac{x}{1-y}\right) = \left(\frac{2\frac{x}{1-y}}{\left(\frac{x}{1-y}\right)^2+1}, \frac{\left(\frac{x}{1-y}\right)^2-1}{\left(\frac{x}{1-y}\right)^2+1}\right) = (x, y),$$

since

$$\frac{2\frac{x}{1-y}}{\left(\frac{x}{1-y}\right)^2+1} = \frac{2(1-y)x}{x^2+(1-y)^2} = \frac{2(1-y)x}{x^2+1-2y+y^2} = \frac{2(1-y)x}{2-2y} = x, \text{ and}$$

$$\begin{aligned} \frac{\left(\frac{x}{1-y}\right)^2-1}{\left(\frac{x}{1-y}\right)^2+1} &= \frac{x^2-(1-y)^2}{x^2+(1-y)^2} = \frac{x^2-1+2y-y^2}{x^2+1-2y+y^2} \\ &= \frac{(1-y^2)-1+2y-y^2}{2-2y} = \frac{2y-2y^2}{2-2y} = \frac{y(1-y)}{1-y} = y. \end{aligned}$$

Thus, we have shown that $F : \mathbb{R} \to \widetilde{S}^1$ is a bijection (1-1 and onto) with inverse $G : \widetilde{S}^1 \to \mathbb{R}$. Hence,

$$(x, y) = F(r) = \left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1} \right) \Leftrightarrow r = G(x, y) = \frac{x}{1-y}.$$

If $(x, y) \in \widetilde{S}^1$ with x and y both rational, then defining $r = \frac{1-y}{x} = G(x, y)$, we have that r is rational and

$$(x, y) = F\left(G(x, y)\right) = F\left(r\right) = \left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1} \right).$$

We know that r is unique since F is 1-1.

Remark. Recall that a triple of positive integers (m, n, p) is a **pythagorean triple** if

$$m^2 + n^2 = p^2 \text{ or } \left(\frac{m}{p}\right)^2 + \left(\frac{n}{p}\right)^2 = 1.$$

The pythagorean triple (m, n, p) is called **primative** if m, n and p are co-prime (i.e., have no common factor other than 1). Note that $\left(\frac{m}{p}, \frac{n}{p}\right)$ is a rational point on the unit circle, and by Problem A, we have

$$\left(\frac{m}{p}, \frac{n}{p}\right) = \left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1} \right)$$

for some rational r, say $r = \frac{a}{b}$. Then

$$\left(\frac{m}{p}, \frac{n}{p}\right) = \left(\frac{2r}{r^2+1}, \frac{r^2-1}{r^2+1} \right) = \left(\frac{2ab}{a^2+b^2}, \frac{a^2-b^2}{a^2+b^2} \right)$$

and

$$\left(\frac{2ab}{a^2+b^2} \right)^2 + \left(\frac{a^2-b^2}{a^2+b^2} \right)^2 = \left(\frac{m}{p}\right)^2 + \left(\frac{n}{p}\right)^2 = 1$$
$$\Rightarrow (2ab)^2 + \left(a^2-b^2\right)^2 = \left(a^2+b^2\right)^2$$
$$\Rightarrow \left(2ab, a^2-b^2, \left(a^2+b^2\right)^2\right) \text{ is a pythagorean triple.}$$

The solution of Problem A can be used to motivate and prove

Theorem. If a and b are positive coprime integers, not both odd, then $(2ab, a^2 - b^2, a^2 + b^2)$ is a primative pythagorean triple. Conversely, let (m, n, p) be a primative pythagorean triple (m, n, p) with m even and n odd, then there is a coprime pair (a, b) of positive integers not both odd, such that

$$(m, n, p) = \left(2ab, a^2 - b^2, a^2 + b^2\right).$$

Solution of Problem B. A game corresponds to a finite sequence of the symbols A and B, where the n-th letter in the sequence is A (resp., B) if Player #1 (resp., Player #2) wins the n-th point of the game. A winning game sequence for Player #1 is one that ends in A, has at least 4 As, and has at least 2 more As than Bs. The probability of such a sequence is obtained by replacing each A by p, each B by $q := 1 - p$, and then multiplying the numbers together. We sum the probabilities of winning game sequences S for Player #1 with lengths less than 7 (i.e., of lengths 4, 5 and 6):

$$\begin{aligned}
&pppp + (qpppp + pqppp + ppqpp + pppqp) \\
&+ (qqpppp + qpqppp + qppqpp + qpppqp + pqqppp \\
&+ pqpqpp + pqppqp + ppqqpp + ppqpqp + pppqqp) \\
&= p^4 + 4qp^4 + 10q^2p^4.
\end{aligned}$$

Note that $10 = \binom{5}{2} = \binom{5}{3}$. A winning sequence for Player #1 of length more than 6, begins with a 6-letter sequence with 3 As and 3 Bs, in any order. There are $\binom{6}{3} = 20$ such sequences, each with probability p^3q^3. Player #1 wins if such sequence is followed by $\mathrm{C}_1 \cdots \mathrm{C}_n$AA, where each C_i is AB or BA and $n \geq 0$. Note that there are 2^n sequences $\mathrm{C}_1 \cdots \mathrm{C}_n$ each with probability $(pq)^n$. Thus, the sum of the probabilities of winning sequences for player #1 of length greater than 6 is

$$20p^3q^3\left(\sum_{n=0}^{\infty}(2^n(pq)^n p^2\right) = 20p^5q^3\left(\sum_{n=0}^{\infty}(2pq)^n\right) = \frac{20p^5q^3}{1-2pq}.$$

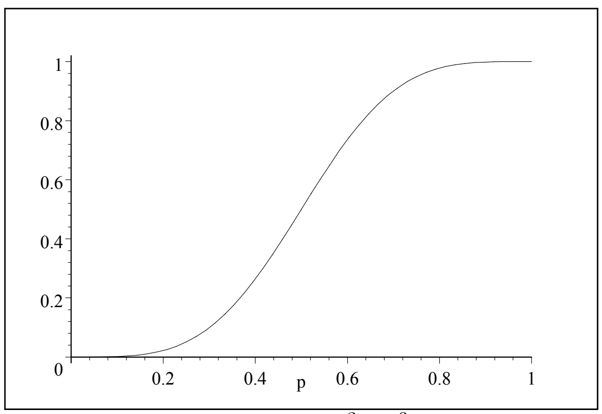
Note that the geometric series is convergent since

$$2pq = 2p(1-p) = \tfrac{1}{2} - 2(p - \tfrac{1}{2})^2 \leq \tfrac{1}{2} < 1.$$

The sum of the probabilities of all the winning sequences for player #1 is then

$$
\begin{aligned}
&p^4 + 4qp^4 + 10q^2p^4 + \frac{20p^5q^3}{1-2pq} = p^4(1 + 4q + 10q^2 + \frac{20pq^3}{1-2pq}) \\
&= p^4\left(\frac{(1+4q+10q^2)(1-2pq)+20pq^3}{1-2pq}\right) = p^4\left(\frac{1-2pq+4q-8pq^2+10q^2}{1-2pq}\right) \\
&= p^4\left(\frac{1+(4-2p)q+(10-8p)q^2}{1-2pq}\right) = p^4\left(\frac{1+(4-2p)(1-p)+(10-8p)(1-p)^2}{1-2p(1-p)}\right) \\
&= p^4\frac{15-34p+28p^2-8p^3}{1-2p+2p^2}.
\end{aligned}
$$

We graph this function of p:



$p^4\frac{15-34p+28p^2-8p^3}{1-2p+2p^2}$