Problems of the Month for UH Mānoa Undergraduates
Solutions of Problems for August 2007

Easier Problem. Ten points are randomly placed in a disk. Show that the probability is less than 50% that a 60° wedge of the disk (with vertex at the center) can be cut out so that at least five of the ten random points are in this wedge. For definiteness, you may assume that the two straight edges of the wedge are included in the wedge, but it makes no difference in the probability. Note that the wedge is not chosen prior to randomly placing the points. Experimentation suggests that the desired probability is considerably less than 50%. Giving a proof of that will augment the award by an unspecified amount.

Harder Problem. Recall that for nonnegative integers k and m with $k \leq m$ the binomial coefficient $\binom{m}{k}$ is defined by

$$\binom{m}{k} := \frac{m!}{k!(m-k)!} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1}$$

if $k \neq 0$, and $\binom{m}{0} := 1$. Let p, q and n be nonnegative integers with $p+q \leq n$. Prove that

$$\binom{n-q}{p}\left(\frac{\binom{q}{0}}{\binom{n}{p}} + \frac{\binom{q}{1}}{\binom{n}{p+1}} + \frac{\binom{q}{2}}{\binom{n}{p+2}} + \cdots + \frac{\binom{q}{q}}{\binom{n}{p+q}}\right) = \frac{n+1}{n+1-q}.$$

Hint. Let $f(p,q,n)$ be the expression in the large parentheses. First show that $\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}$ for $1 \leq k \leq m$. Apply this to all but the first and last numerators in $f(p,q,n)$. Deduce that $f(p,q,n) = f(p,q-1,n) + f(p+1,q-1,n)$ for $q \geq 1$. Now, for fixed p and n, use mathematical induction on q, noting that the base case $q = 0$ clearly holds.

Rules

The following rules may be changed or clarified from month to month.

1. Any "regular" undergraduate currently enrolled at UH Manoa is eligible to compete.

2. Write a complete solution with all details to either problem or both.

3. Submit your solution(s) *electronically* before the end of the above month to

bleecker@math.hawaii.edu

For the subject line of your email use "problem of the month". Either write your solution within the body of your email or within attachment(s) in the form of readable pdf files or images of your work in jpg format (e.g., scanned or digitally photographed).

4. Solutions will be judged by a committee of professors according to a combination of criteria: accuracy, attention to details, chronological order of submission, and neatness, but not necessarily in that order.

5. Before the end of the 10-th day of the month that follows, the winner(s) will be announced on the Department web site. Moreover, winners will receive a (modest) cash prize of $10 for the easier problem and $25 for the harder problem, as soon as the checks can be extracted from the Hanf Fund at the UH Foundation, a process that may take several weeks.