# Problems of the Month for UH Mānoa Undergraduates
## Solutions of Problems for August 2007

**Easier Problem**. Ten points are randomly placed in a disk. Show that the probability is less than 50% that a $60^\circ$ wedge of the disk (with vertex at the center) can be cut out so that at least five of the ten random points are in this wedge. For definiteness, you may assume that the two straight edges of the wedge are included in the wedge, but it makes no difference in the probability.

**Harder Problem**. Recall that for nonnegative integers $k$ and $m$ with $k \le m$ the binomial coefficient $\binom{m}{k}$ is defined by

$$\binom{m}{k} := \frac{m!}{k!(m-k)!} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1}$$

if $k \neq 0$, and $\binom{m}{0} := 1$. Let $p, q$ and $n$ be nonnegative integers with $p+q \le n$. Prove that

$$\binom{n-q}{p}\left(\frac{\binom{q}{0}}{\binom{n}{p}} + \frac{\binom{q}{1}}{\binom{n}{p+1}} + \frac{\binom{q}{2}}{\binom{n}{p+2}} + \cdots + \frac{\binom{q}{q}}{\binom{n}{p+q}}\right) = \frac{n+1}{n+1-q}$$

**Hint**. Let $f(p,q,n)$ be the expression in the large parentheses. First show that $\binom{m}{k} = \binom{m-1}{k-1} + \binom{m-1}{k}$ for $1 \le k \le m$. Apply this to all but the first and last numerators in $f(p,q,n)$. Deduce that $f(p,q,n) = f(p,q-1,n) + f(p+1,q-1,n)$ for $q \ge 1$. Now, for fixed $p$ and $n$, use mathematical induction on $q$, noting that the base case $q = 0$ clearly holds.

**Solution of Easier Problem**. For $n \in \{1, 2, \ldots, 10\}$, let $E_n(k)$ be the event that the $60^\circ$ closed wedge running clockwise from the edge containing the point $x_n$ has exactly $k$ points (including $x_n$) in it. The probability of $E_n(k)$ is given by

$$\begin{aligned} p(E_n(k)) &= \frac{9!}{(9-(k-1))!\,(k-1)!}(\frac{1}{6})^{k-1}(\frac{5}{6})^{9-(k-1)} \\ &= \frac{9!}{(9-k+1)!\,(k-1)!}(\frac{1}{6})^{k-1}(\frac{5}{6})^{9-k+1}. \end{aligned}$$

Let $E_n = E_n(5) \cup \cdots \cup E_n(10)$, a union of a *disjoint* collection of events. Thus,

$$
\begin{aligned}
p\,(E_n) &= \sum\nolimits_{k=5}^{10} p(E_n(k)) \\
&= \sum\nolimits_{k=5}^{10} \left( \frac{9!}{(9-k+1)!\,(k-1)!} (\frac{1}{6})^{k-1} (\frac{5}{6})^{9-k+1} \right) = \frac{241\,973}{5038\,848} = 0.0480214922\ldots.
\end{aligned}
$$

Let $E$ be the event that there is a closed wedge of the disk, with a central angle of $60^\circ$, that can be cut out so that at least five of the ten points are in this wedge. By rotating this wedge in the clockwise direction until the trailing edge of the wedge contacts one of the 10 points, we deduce that if $E$ occurs then one (or more) of the events $E_n$ occurs. Conversely, if one of the $E_n$ occurs, then $E$ clearly occurs as well. In other words,

$$E = E_1 \cup \cdots \cup E_{10}.$$

Although the collection $\{E_1, \ldots, E_{10}\}$ is not disjoint (i.e., $E_i \cap E_j \neq \phi$ for $i \neq j$), we still have

$$P(E) \leq p(E_1) + \cdots + p(E_{10}) = 10p\,(E_n) = \frac{241\,9730}{5038\,848} = 0.480214922\ldots < 50\%.$$

A program in C can simulate the process. After 1,000,000 iterations, 311269 had 5 points within a 60 degree arc, resulting in a probability of 31.126900%. The program submitted by the winner is as follows:

```
#include <math.h>
#define angle
main ()
{
float randoms[10]; float temp;
int count, iterations, chance, lowest, sort, repeat;
chance=0;
printf(”How many iterations do you wish to run through?”); scanf(”%d”,&repeat);
srand((unsigned)(time(0)));
for(iterations=0; iterations<repeat; iterations++){
for(count=0; count<10; count++) {randoms[count]=6.0*rand()/(float)(0x7fffffff);}
/*this section prints lists for debugging purposes*/ /*printf(”The list
of numbers is: \n”); for(count=0; count<10; count++) printf(”\t%f\n”,
randoms[count]);*/
```

```
/*this section sorts the random-list */
for(count=0; count<10; count++)
{
lowest=count;
for(sort=count; sort<10; sort++) {if(randoms[sort]<randoms[lowest]) lowest=sort;}
temp=randoms[lowest]; randoms[lowest]=randoms[count]; randoms[count]=temp;
}
/*this section prints the sorted list for debugging purposes*/ /*printf(”The sorted list of numbers is: \n”); for(count=0; count<10; count++) printf(”\t%f\n”, randoms[count]);
/*This section checks for closeness*/
temp=0;
for(count=0; count<10; count++) {if ( fmodf( fabsf(randoms[(count+4)%10]-randoms[count]), 5.0)<=1) {temp=1; count=10;}}
if (temp==1) {chance++;
/*printf(”Success! %d of %d\n”, chance, iterations);*/
}
}
printf(”After %d iterations, %d had 5 points within a 60degree arc, resulting in a probability of %fpercent.\n\n\n”, iterations, chance, (float)100*chance/iterations);
}
```

**Solution of Harder Problem**. First, for $k > 1$

$$\binom{m-1}{k-1} + \binom{m-1}{k} = \frac{(m-1)!}{(m-k)!\,(k-1)!} + \frac{(m-1)!}{(m-1-k)!k!}$$

$$= \frac{k\,(m-1)!}{(m-k)!k\,(k-1)!} + \frac{(m-k)\,(m-1)!}{(m-k)\,(m-1-k)!k!}$$

$$= \frac{k\,(m-1)! + (m-k)\,(m-1)!}{(m-k)!k!} = \frac{k\,(m-1)! + (m\,(m-1)! - k\,(m-1)!)}{(m-k)!k!}$$

$$= \frac{m\,(m-1)!}{(m-k)!k!} = \frac{m!}{(m-k)!k!} = \binom{m}{k}.$$

Applying this, we get

$$
\begin{aligned}
f(p,q,n) &= \frac{\binom{q}{0}}{\binom{n}{p}} + \frac{\binom{q}{1}}{\binom{n}{p+1}} + \frac{\binom{q}{2}}{\binom{n}{p+2}} + \cdots + \frac{\binom{q}{q-1}}{\binom{n}{p+q-1}} + \frac{\binom{q}{q}}{\binom{n}{p+q}} \\
&= \frac{\binom{q-1}{0}}{\binom{n}{p}} + \frac{\binom{q-1}{0} + \binom{q-1}{1}}{\binom{n}{p+1}} + \frac{\binom{q-1}{1} + \binom{q-1}{2}}{\binom{n}{p+2}} + \cdots \\
&+ \frac{\binom{q-1}{q} + \binom{q-1}{q-1}}{\binom{n}{p+q-1}} + \frac{\binom{q-1}{q-1}}{\binom{n}{p+q}} \\
&= \frac{\binom{q-1}{0}}{\binom{n}{p}} + \frac{\binom{q-1}{1}}{\binom{n}{p+1}} + \frac{\binom{q-1}{2}}{\binom{n}{p+2}} + \cdots + \frac{\binom{q-1}{q-1}}{\binom{n}{p+q-1}} \\
&+ \frac{\binom{q-1}{0}}{\binom{n}{p+1}} + \frac{\binom{q-1}{1}}{\binom{n}{p+2}} + \frac{\binom{q-1}{2}}{\binom{n}{p+3}} + \cdots + \frac{\binom{q-1}{q-1}}{\binom{n}{p+q}} \\
&= f(p,q-1,n) + f(p+1,q-1,n).
\end{aligned}
$$

Assuming the desired result for $q-1$ then yields

$$
\begin{aligned}
&\binom{n-q}{p} f(p,q,n) = \binom{n-q}{p} f(p,q-1,n) + \binom{n-q}{p} f(p+1,q-1,n) \\
&= \frac{\binom{n-q}{p}}{\binom{n-(q-1)}{p}} \binom{n-(q-1)}{p} f(p,q-1,n) \\
&+ \frac{\binom{n-q}{p}}{\binom{n-(q-1)}{p+1}} \binom{n-(q-1)}{p+1} f(p+1,q-1,n) \\
&= \frac{\binom{n-q}{p}}{\binom{n-(q-1)}{p}} \left( \frac{n+1}{n+1-(q-1)} \right) + \frac{\binom{n-q}{p}}{\binom{n-(q-1)}{p+1}} \left( \frac{n+1}{n+1-(q-1)} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\binom{n-q}{p}}{\binom{n-(q-1)}{p}} \left(\frac{n+1}{n-q+2}\right) + \frac{\binom{n-q}{p}}{\binom{n-(q-1)}{p+1}} \left(\frac{n+1}{n-q+2}\right) \\
&= \frac{(n-q)\cdots(n-q-p+1)}{(n-q+1)\cdots(n-q+1-p+1)} \left(\frac{n+1}{n-q+2}\right) \\
&+ \frac{(p+1)\cdot(n-q)\cdots(n-q-p+1)}{(n-q+1)\cdots(n-q+1-(p+1)+1)} \left(\frac{n+1}{n-q+2}\right) \\
&= \frac{(n-q-p+1)}{(n-q+1)} \left(\frac{n+1}{n-q+2}\right) + \frac{(p+1)}{(n-q+1)} \left(\frac{n+1}{n-q+2}\right) \\
&= \left(\frac{(n-q-p+1)}{(n-q+1)} + \frac{(p+1)}{(n-q+1)}\right) \left(\frac{n+1}{n-q+2}\right) \\
&= \left(\frac{n-q+2}{n-q+1}\right) \left(\frac{n+1}{n-q+2}\right) = \frac{n+1}{n+1-q},
\end{aligned}
$$

and the induction on $q$ is complete.